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Not Ignoring is not Knowing

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Abstract

Recent debates in epistemology put forward the idea that ignorance should not be analyzed in terms of knowledge, but rather as an independent epistemic notion. On the basis of this analysis, we propose a new approach to the problem of logical omniscience in terms of *not ignoring* rather than *knowing*. We motivate this approach from an epistemological perspective and show how it allows to avoid logical omniscience already in usual possible worlds semantics. Finally, we discuss the problematic case of a stronger form of omniscience arising in this setting.

Keywords: ignorance, logical omniscience, epistemic logic, knowledge representation

Introduction

In the literature on epistemic logic and knowledge representation, ignorance is traditionally considered as an ancillary concept with respect to knowledge. More precisely, ignorance coincides with non-knowledge, and knowledge with non-ignorance, which means that the notion of ignorance is a complement to the notion of knowledge. Such a view is endorsed by Zimmerman [32], Driver [5], Fields [12], Haack [14], Le Morvan [20, 21, 22] and is today dubbed *Standard view* [23]. We challenge this view by showing some of its limitations with respect to an instance of the problem of logical omniscience affecting traditional epistemic logic. We show how such limitations can be partly overcome by modifying the definition of ignorance and taking it as a primitive notion.

The article is organized as follows. In the first section, we present the Standard view and some of its limitations. More precisely, we focus on the so-called problem of logical omniscience, by showing how an instance of this problem threatens this view. In the second section, we present an alternative view, called *Logical view*, and show that a logical system based on this view does not suffer from the problem of logical omniscience. In the third and final section, we discuss some of the shortcomings of the Logical view with respect to a stronger form of logical omniscience and analyze some possible improvements.

1 Ignorance and omniscience: the Standard view

According to the Standard view (henceforth SV), an adequate epistemic analysis of ignorance can be provided in terms of non-knowledge. As a consequence, all the theoretical apparatus used to deal with knowledge can be employed to analyze ignorance. In particular, one can adopt the possible worlds semantics proposed by Hintikka [15, 16, 17] to modelling epistemic logic as a basis for ignorance representation. The intuitive idea beyond possible worlds semantics is that an agent knows a fact ϕ if ϕ is true in all the worlds (or states) she thinks possible.

Formally, the set of formulas of an epistemic propositional language $\mathcal{L}^K$ is defined by:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid K\phi$$

where $K\phi$ represents the fact that an agent knows that ϕ. Other propositional operators can be defined in a standard way: $\phi \vee \psi \Leftrightarrow \neg(\neg\phi \wedge \neg\psi)$, $\phi \rightarrow \psi \Leftrightarrow \neg\phi \vee \phi$, $\phi \leftrightarrow \psi \Leftrightarrow (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$.

The semantics of this language is provided by Kripke semantics which formalizes the intuitive ideas behind possible worlds.

Definition 1.1 (Frames, Models, Satisfaction) *A Kripke Frame $F = \langle W, R\rangle$ is a tuple where W is a set of epistemic alternatives for the agent, and $R \subseteq W \times W$ is an accessibility relation. A Kripke Model $M = \langle F, v\rangle$, is a tuple where F is a Kripke frame and $v : P \rightarrow 2^W$ is an interpretation for a set of propositional variables P. Given a model M and a formula ϕ, we say that ϕ is true in M at world w, written $M, w \models_K \phi$ if:*

1. *$M, w \models_K p$ if $w \in v(P)$,*
2. *$M, w \models_K \neg\phi$ if it is not the case that $M, w \models_K \phi$,*
3. *$M, w \models_K \phi \wedge \psi$ if $M, w \models_K \phi$ and $M, w \models_K \psi$,*
4. *$M, w \models_K K\phi$ if for all w', such that Rww', $M, w' \models_K \phi$.*

We say that ϕ is valid on M and write $M \models_K \phi$, if $M, w \models_K \phi$ for all w in W. If for all M based on F we have $M \models_K \phi$, we say that ϕ is valid on F and write $F \models_K \phi$.

The system characterized by this semantics is called system **K** and it is defined by the following axioms and rules:

($TAUT$) All instances of propositional tautologies

(K) $K(\phi \to \psi) \to (K\phi \to K\psi)$

(NR) From $\vdash_K \phi$ infer $\vdash_K K\phi$

(MP) Modus Ponens

(Sub) Substitution of equivalences

A derivation of **K** is a finite sequence of $\mathcal{L}^K$-formulas such that each formula is either the instantiation of an axiom or the result of applying an inference rule to previous formulas in the sequence. A formula $\phi \in \mathcal{L}^K$ is called a theorem, noted $\vdash_K \phi$, if it occurs in a derivation of **K**. Clearly, the necessitation rule NR and the axiom scheme K are valid on all frames of Kripke semantics.

An advantage of SV is that the use of Kripke semantics makes possible an epistemic analysis of the notion of ignorance. However, one of the consequences of adopting this approach is that all the relevant problems in the analysis of the notion of knowledge become problematic also for the notion of ignorance. One very well known example is what was dubbed the problem of *logical omniscience* by Hintikka [18]. Intuitively, this problem points to the fact that the agents modelled by standard Kripke semantics are perfect reasoners, since they know all the consequences of their knowledge and, in particular, they know all the tautologies. This fact becomes problematic when one wants to represent the knowledge of real-world agents (such as human beings or computers). In this case there are practical limitations to the idealized capacities of reasoning of the agent. Hence the problem of logical omniscience is the problem to find a balance between the representation of real-world agents and the representation of the logical capacities of such agents. There exist many instances of logical omniscience, see Fagin *et al.* [7] for a survey. The problem can be formally stated as follows:

- Logical omniscience problem (Hintikka [18]):

$$\text{if } \phi \to \psi \text{ and } K\phi, \text{ then } K\psi. \qquad (LO)$$

Syntactically, LO can be proved as follows:

1. $\phi \to \psi$ (assumption)
2. $K\phi$ (assumption)
3. $K(\phi \to \psi)$ (NR, 1.)

4. $K(\phi \to \psi) \to (K\phi \to \psi)$ (ax. sch. K)

5. $K\phi \to K\psi$ (MP, 4., 3.)

6. $K\psi$ (MP, 5., 2.)

It is also easy to provide the semantic proof of the validity of LO on all frames:

Proof. (Semantic proof of LO) Let (i) $M \models_K \phi \to \psi$, (ii) $M \models_K K\phi$ and (iii) $M \not\models_K K\psi$. Thus, there exists a w, such that $M, w \not\models_K K\psi$, which means, by def. 1.1, that (v) there exists a world w', such that Rww' and $M, w' \models_K \neg\psi$. By (ii), we have $M, w \models_K K\phi$ that is (vi) for all w', such that Rww', $M, w' \models_K \phi$. From (i) and (vi) we obtain that for all w', such that Rww', $M, w' \models_K \psi$, that contradicts (v). Thus, if $M \models_K \phi \to \psi$ and $M \models_K K\phi$, then $M \models_K K\psi$.

■

The problem of logical omniscience is usually formulated in terms of knowledge as follows:

If an agent *knows* ϕ, she *knows* all the consequences of ϕ. (LO_K)

However, LO_K can be naturally reformulated in terms of ignorance.

If an agent *does not ignore* ϕ, she *does not ignore* all the consequences of ϕ. (LO_I)

By considering ignorance to be the complement of knowledge, SV identifies LO_K and LO_I. From this perspective, knowing corresponds to not ignoring and ignoring corresponds to not knowing. As a consequence, SV is confronted with a form of logical omniscience, both for knowledge and ignorance.

Among the solutions proposed to solve LO_K there are:

- Impossible worlds semantics (Cresswell [2, 3, 4], Hintikka [18], Rantala [27], Wansing [31], Fagin *et al.* [7]);
- Non-classical worlds semantics (Levesque [24], Lakemeyer [19], Fagin *et al.* [7, 8]);
- Awareness (Fagin and Halpern [6]).

All these strategies share a common feature: they all modify Kripke semantics in order to deal with LO_K.[1] In particular, according to def. 1.1, all the

[1]The so-called syntactic approaches also reject Kripke semantics as a basis for solving LO_K.

worlds $w \in W$ are consistent, thus it is not possible to have $\phi, \neg\phi \in w$. The impossible worlds semantics change this standard aspect of Kripke semantics, and defines (at least some) worlds, called the impossible worlds, as inconsistent. The use of non-classical worlds presupposes the change of the underlying logic from classical (that is used for non-modal operators definitions, as in def. 1.1) to some weaker non-classical logics. The use of awareness function implements the change of the definition of K-operator by adding a new condition, the awareness condition. In the next section, we show that such modifications are not strictly necessary and that a different analysis of ignorance allows one to keep a standard Kripke semantics while dealing with logical omniscience.

2 Ignorance and omniscience: the Logical view

In the previous section, we have seen that SV suffers from LO in the same way as standard epistemic logic does. However, besides LO, there are independent reasons for doubting that SV provides a correct analysis of ignorance. According to SV, whenever an agent does not know p, she is ignorant about p. Moreover, as most epistemologists agree, knowledge is factive, i.e. if an agent knows p, p is true. The direct consequence of these two assumptions is that an agent is ignorant about all false propositions. This is a bold position about the cognitive capacities of an agent and sometimes is just counter-intuitive. For instance, let us consider an agent who knows that the capital of Brazil is Brasília. By knowing this, it is clear that the agent does not know that the capital of Brazil is Paris. According to SV, this is the same to say that the agent is ignorant about the fact that the capital of Brazil is Paris. However, it seems odd to state that the agent is ignorant about the fact that the capital of Brazil is Paris on the basis of her knowledge that the capital is Brasília.

Another line of criticism to SV concerns the ignorance in the so-called *Gettier-cases* (see [13]). Let us borrow an example from [23], p. 26-27. Consider an agent who looks at the clock which tells her that it is 7 PM. The agent believes that it is 7 PM, because she trusts her clock. However, the clock has stopped exactly 24 hours ago, but the agent does not know this. Gettier-cases are used to show that an agent can have a justified true belief in a proposition without knowing this proposition. But can one be sure that these cases are cases of ignorance? One may say that the agent in the example is ignorant about the fact that the clock has stopped, that it is unreliable etc., but it is not plausible that she is ignorant that it is 7 PM. Other arguments against SV can be found in [23].

In the literature on epistemic logic, it is possible to find an alternative definition of ignorance. In this case, ignorance corresponds to neither knowing some proposition, nor knowing its negation. In other words, it corresponds to *not knowing whether*. This view is dubbed Logical view (henceforth LV) by Fan [11]. Such definition was used by van der Hoek and Lomuscio [29, 30] in order to introduce a logic for ignorance. This logic, called **Ig**, contains an ignorance operator I as the sole primitive modality. To the best of our knowledge, **Ig** is the first logical system explicitly formalizing ignorance independently from knowledge. The set of formulas of the epistemic propositional language $\mathcal{L}^I$ of **Ig** is defined by the following grammar.

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid I\phi$$

As before, the other operators ($\vee$, $\rightarrow$, and $\leftrightarrow$) can be defined in a standard way.

The semantics of this language is provided by Kripke semantics. The only modification with respect to definition 1 is the last clause, where the condition on K is replaced with:

4'. $M, w \models_{Ig} I\phi$ if there exist w', w'' such that Rww', Rww'', $M, w' \models_{Ig} \phi$, and $M, w'' \models_{Ig} \neg\phi$.

Syntactically, the modal system **Ig** is characterized by the following axioms and rules.

($TAUT$) All instances of propositional tautologies.

($I1$) $I\phi \leftrightarrow I\neg\phi$

($I2$) $I(\phi \wedge \psi) \rightarrow (I\phi \vee I\psi)$

($I3$) $((\neg I\phi \wedge I(\alpha_1 \wedge \phi)) \wedge \neg I(\phi \rightarrow \psi) \wedge I(\alpha_2 \wedge (\phi \rightarrow \psi))) \rightarrow (\neg I\psi \wedge I(\alpha_1 \wedge \psi))$

($I4$) $(\neg I\psi \wedge I\alpha) \rightarrow (I(\alpha \wedge \psi) \vee I(\alpha \wedge \neg\psi))$

(RI) From $\vdash_{Ig} \phi$ infer $\vdash_{Ig} \neg I\phi \wedge (I\alpha \rightarrow I(\alpha \wedge \phi))$

(MP) Modus Ponens

(Sub) Substitution of equivalences

A derivation of ϕ from Γ, noted $\Gamma \vdash_{Ig} \phi$, is a finite sequence of $\mathcal{L}^I$-formulas such that each formula is either the instantiation of an axiom, or an element in Γ, or follows from the previous formulas in the sequence by an inference rule. A derivation of ϕ is a derivation of ϕ from the empty set. We write $\vdash_{Ig} \phi$ if there is a derivation of ϕ in **Ig**.

Van der Hoek and Lomuscio showed that **Ig** is sound and complete, i.e.

Theorem 2.1 (van der Hoek and Lomuscio) *Given system* ***Ig****, for any formula* ϕ *we have the following:* $\vdash_{\boldsymbol{Ig}} \phi$ *iff* $\models \phi$*.*

By exploiting the I operator, it is now possible to formalize LO_I independently from the K operator:

$$\text{if } \phi \to \psi \text{ and } \neg I\phi, \text{ then } \neg I\psi. \qquad (LO_I^W)$$

It is easy to verify that in the system **Ig** the form of omniscience encapsulated in LO_I^W fails.

Theorem 2.2 *(LO_I^W) is not valid in* ***Ig****.*

Proof. We construct a counter-model M', such that $M' \models_{Ig} p \to q$, $M' \models_{Ig} \neg Ip$ and $M' \not\models_{Ig} \neg Iq$. $M' = \langle W', R', v\rangle$, where $W' = \{w, w', w''\}$, $R' = \{(w, w'), (w, w'')\}$ and $v(p) = \{w\}$, $v(q) = \{w, w''\}$. Graphically this model can be represented as in figure 1. It is evident, that $M' \models_{Ig} p \to q$, because all worlds containing p also contain q; $M', w \models_{Ig} \neg Ip$, because in all accessible worlds from w we have $\neg p$; $M', w' \models_{Ig} \neg Ip$ and $M', w'' \models_{Ig} \neg Ip$, because there are no accessible worlds from w' and w''; and $M', w \not\models_{Ig} \neg Iq$, because there exists a world w' accessible from w, containing $\neg q$ and there exists a world w'' accessible from w, containing q.

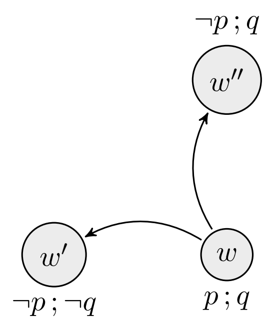

Figure 1: Model M'

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The reason for the failure of this form of logical omniscience in **Ig** is that for all worlds of our counter-model M' there are no accessible worlds where p is true. Thus, the model represents a situation where $p \to q$ and the agent is not ignorant about p in the sense that she considers p to be false. One cannot construct a counter-model to LO_I^W where the agent considers p as a true proposition, because in this case q will belong to all the worlds containing p and thus will not be ignored.

3 A stronger form of Logical Omniscience

We showed that the form of logical omniscience incapsulated in the principle LO_I^W is not derivable in **Ig**. However, by taking into account an agent who is not ignorant about ϕ and considers ϕ as a true proposition, i.e., $\neg I\phi \wedge \phi$, it is possible to derive a stronger form of omniscience which cannot be avoided in **Ig**. From an epistemological perspective, an account of these situations of ignorance seems a reasonable desideratum.

Let us formalize this stronger version of LO_I as follows:

$$\text{if } \phi \rightarrow \psi \text{ and } \neg I\phi \wedge \phi, \text{ then } \neg I\psi. \qquad (LO_I^S)$$

The principle LO_I^S can be also formalized as "if $\phi \rightarrow \psi$ and $\neg I\phi \wedge \phi$, then $\neg I\psi \wedge \psi$". The results about LO_I^S proved in this section hold also for this alternative formulation.

Theorem 3.1 *(LO_I^S) is valid in **Ig**.*

Proof. Let (i) $M \models_{Ig} \phi \rightarrow \psi$, (ii) $M \models_{Ig} \neg I\phi \wedge \phi$ and (iii) $M \not\models_{Ig} \neg I\psi$. Thus, from (ii) we have (iv) for all w' such that Rww' $M, w' \models_{Ig} \phi$. According to (iii) there is a world w, such that $M, w \models_{Ig} I\psi$, i.e. there exist w'' and w''' such that Rww'', Rww''', $M, w'' \models_{Ig} \psi$ and $M, w''' \models_{Ig} \neg\psi$. The existence of a world w''' such that $M, w''' \models_{Ig} \neg\psi$ together with (i) and (iv) entails a contradiction.

■

The fact that LO is valid in standard Kripke semantics, while LO_I^W is not, allowed us to put forward the idea that LO_I^W is weaker than LO. From this perspective, one may wonder what is the relationship between LO and LO_I^S, as they are both valid in Kripke frames. In the reminder of this section we show that LO_I^S is stronger than LO. To prove this result, Kripke semantics is insufficient as it does not allow us to distinguish these two principles. Therefore, we exploit neighborhood semantics, a formalism particularly well-behaved for non-normal modal logics (see Dana Scott [28] and Richard Montague [25]).

We use again the language $\mathcal{L}^K$. The neighborhood semantics for this language is defined as follows.

Definition 3.2 (Neighborhood Semantics) *A neighborhood model is a tuple $M = \langle W, N, V\rangle$, where W is a nonempty set of possible worlds called the domain, N is a mapping from W to sets of subsets of W ($N_w \subseteq \mathfrak{P}(W)$ for each world w in W), and V is a function that associates each propositional formula with a subset of W. A neighborhood frame is a neighborhood model without valuation. Given a neighborhood model M and a formula ϕ, we say that ϕ is true in M at world w, written $M, w \models_E \phi$ if:*

1. *$M, w \models_E p$ if $w \in V(p)$*
2. *$M, w \models_E \neg\phi$ if $M, w \not\models_E \phi$*
3. *$M, w \models_E \phi \wedge \psi$ if $M, w \models_E \phi$ and $M, w \models_E \psi$*
4. *$M, w \models_E K\phi$ if $\|\phi\|^M \in N_w$*

where $\|\phi\|^M$ denotes the truth set of ϕ in M, that is $\|\phi\|^M = \{w \text{ in } M : M, w \models \phi\}$.

The minimal system **E** is defined as follows (see Chellas [1]).

($TAUT$) All instances of propositional tautologies

(RE) From $\vdash_E \phi \leftrightarrow \psi$ infer $\vdash_E K\phi \leftrightarrow K\psi$

(MP) Modus Ponens

(Sub) Substitution of equivalences

A derivation of ϕ from Γ, noted: $\Gamma \vdash_E \phi$, is a finite sequence of $\mathcal{L}^K$-formulas such that each formula is either the instantiation of an axiom, or an element in Γ, or follows from the previous formulas in the sequence by an inference rule. A derivation of ϕ is a derivation of ϕ from the empty set. We write $\vdash_E \phi$ if there is a derivation of ϕ in **E**.

Theorem 3.3 *Given system **E**, for any formula ϕ we have the following: $\vdash_E \phi$ iff $\models_E \phi$.*

It is well known that LO fails in the system **E**.

Theorem 3.4 *LO is not valid in **E**.*

Proof. We construct a counter-model M^*, such that $M^* \models_E p \rightarrow q$, $M^* \models_E Kp$ and $M^* \not\models_E Kq$. $M^* = \langle W^*, N^*, V^* \rangle$, where $W^* = \{w, w'\}$, $N^*_w = N^*_{w'} = \{\{w\}\}$, $V^*(p) = \{w\}$ and $V^*(q) = \{w, w'\}$. Graphically this model can be represented as in figure 2. It is clear that $M^* \models_E p \rightarrow q$, because each world w that contains p also contains q; $M^* \models_E Kp$, because for each world w, its neighborhood N^*_w contains the truth set of p, that is $\{w\}$; and $M^* \not\models_E Kq$, because there exists a world w, such that its neighborhood N^*_w does not contain the truth set of q, that is $\{w, w'\}$.

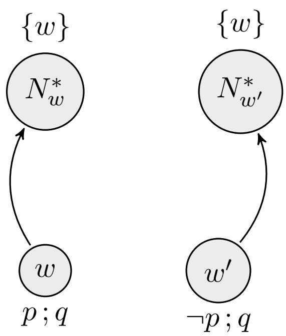

Figure 2: Model M^*

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To compare LO, LO_I^W and LO_I^S, let us interpret the I operator in neighborhood semantics. In order to do so, we exploit known results on non-contingency logic. Indeed, the system **Ig** was introduced as an epistemic interpretation of a non-contingency logic. In this logic, contingency of some formula ϕ means that ϕ is possibly true and possibly false, and non-contingency of ϕ means that ϕ is necessarily true or necessarily false. As noticed by Fan *et al.* [10], ignorance can be interpreted as an epistemic counterpart of contingency. As a consequence, one can adopt the neighborhood semantics provided in Fan and van Ditmarsch [9] to modelling ignorance in accordance with LV.

The system of Fan and van Ditmarsch, called **CCL**, is defined on a propositional language $\mathcal{L}^{\Delta}$:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid \Delta\phi$$

The language $\mathcal{L}^{\Delta}$ is interpreted via neighborhood semantics. The only modification with respect to definition 3.2 is the last clause, where the condition on K is replaced with:

4*. $M, w \models_{CCL} \Delta\phi$ if $\|\phi\|^M \in N_w$ or $\|\neg\phi\|^M \in N_w$

The system **CCL** is defined as follows.

($TAUT$) All instances of propositional tautologies.

($\Delta \leftrightarrow$) $\Delta\phi \leftrightarrow \Delta\neg\phi$

($RE\Delta$) From $\vdash_{CCL} \phi \leftrightarrow \psi$ infer $\vdash_{CCL} \Delta\phi \leftrightarrow \Delta\psi$

(MP) Modus Ponens

(Sub) Substitution of equivalences

A derivation of ϕ from Γ, noted: $\Gamma \vdash_{CCL} \phi$, is a finite sequence of $\mathcal{L}^{\Delta}$-formulas such that each formula is either the instantiation of an axiom, or an element in Γ, or follows from the previous formulas in the sequence by an inference rule. A derivation of ϕ is a derivation of ϕ from the empty set. We write $\vdash_{CCL} \phi$ if there is a derivation of ϕ in **CCL**.

Fan and van Ditmarsch prove soundness and completeness for **CCL**, i.e.,

Theorem 3.5 (Fan and van Ditmarsch) *Given system **CCL**, for any formula ϕ we have the following: $\vdash_{CCL} \phi$ iff $\models_{CCL} \phi$.*

Let us define the I operator as the negation of Δ: $I\phi \Leftrightarrow \neg\Delta\phi$. Consequently, $\neg I\phi \Leftrightarrow \neg\neg\Delta\phi \Leftrightarrow \Delta\phi$. This permits us to check the validity of LO_I^W and LO_I^S with respect to neighborhood frames.

Theorem 3.6 *LO_I^W is invalid in **CCL**.*

Proof. To prove this theorem we use the same counter-model M^* as in the proof of the theorem 3.4, that is $M^* = \langle W^*, N^*, V^* \rangle$, where $W^* = \{w, w'\}$, $N^*_w = N^*_{w'} = \{\{w\}\}$, $V^*(p) = \{w\}$ and $V^*(q) = \{w, w'\}$. It is clear that $M^* \models_{CCL} p \to q$ by the same reasons as before; $M^* \models_{CCL} \neg Ip$, that is $M^* \models_{CCL} \Delta p$, because for all worlds w its neighborhood N^*_w contains the truth set of p; and $M^* \not\models_{CCL} \neg Iq$, that is $M^* \not\models_{CCL} \Delta q$, because there exists a world w such that its neighborhood N^*_w does not contain neither the truth set of q, nor the truth set of $\neg q$.

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We have shown that neither LO, nor LO_I^W is valid in **CCL**. The following result permits to distinguish LO_I^S from LO (and LO_I^W); in particular, it shows that the former is stronger than the latter.

Theorem 3.7 *LO_I^S is valid in **CCL**.*

Proof. Let (i) $M \models_{CCL} \phi \to \psi$, (ii) $M \models_{CCL} \neg I\phi \wedge \phi$ and (iii) $M \not\models_{CCL} \neg I\psi$. According to (ii) for all worlds w, (iv) $\phi \in w$ and (v) either $\|\phi\|^M \in N_w$, or $\|\neg\phi\|^M \in N_w$. By (i) and (iv) we have (vi) $\psi \in w$ for all w. From (vi) and (iv) we conclude that the truth sets of ϕ and ψ coincide, as well as the truth sets of $\neg\phi$ and $\neg\psi$. According to (iii) there exists a world w' such that $\|\psi\|^M \notin N_w$ and $\|\neg\psi\|^M \notin N_w$. This contradicts (v).

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Even though LO is not valid in system **E**, as shown by theorem 3.6, the rule RE already contains a disguised form of logical omniscience. RE represents closure under logical equivalence, a necessary property of knowledge for **E**-structures. As noticed in Fagin *et al.* [7], closure under logical equivalence means that "while agents need not know all logical consequences of their knowledge, they are unable to distinguish between logically equivalent formulas" [7], p. 344. Thus, system **E** equipped with neighborhood semantics permits to avoid *almost* all forms of logical omniscience, not all of them. The same remark can be made about **CCL**. The rule $RE\Delta$ is the dual of the rule RE and represents closure under logical equivalence. Therefore, in **CCL**, agents are not ignorant about all logically equivalent formulas.

4 Conclusion

By reformulating the standard logical omniscience problem (LO_K) as LO_I, we proposed a new approach to the problem of logical omniscience in terms of not ignoring rather than knowing. Moreover, our analysis allowed us to distinguish two different forms of logical omniscience inside LO_I, a weaker and a stronger one (LO_I^W and LO_I^S, respectively). We exploited both Kripke and neighbourhood semantics to show that LO_K is stronger than LO_I^W, but weaker than LO_I^S. We take these results to contribute to the epistemological debate on the nature of ignorance. On the one hand, SV suffers from all forms of logical omniscience because it is unable to distinguish between LO_K and LO_I. On the other, LV can avoid at least the weaker problem encapsulated by LO_I^W, thus offering a less problematic definition and formalization of the notion of ignorance. Nonetheless, the logical systems representing ignorance in accordance with LV (i.e. **Ig** and **E**) still suffer from LO_I^S. To avoid the validity of LO_I^S, at least two ways seem open: modifying the semantics, or changing the definition of the I operator. The former would amount to a rather classical strategy to solve the problem of logical omniscience, while the latter points to a more original solution. Another possible criticism of **Ig** and **E** is that they avoid LO_K trivially, simply because their languages do not contain the K operator. It would be interesting to introduce a multi-modal system in which K and I operators can be independently defined and test such a system on all the different forms of logical omniscience we presented. We leave these tasks for future investigations.

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