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Numerical Existence Property and Numerical Content: Intuitionistic Logic versus Arithmetical Logic

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Abstract

I examine claims of numerical existence for the intuitionistic disjunction and existential quantifier. I argue that those claims do not secure numerical content and that a polynomial translation of logical constants comes closer to a numerical language for mathematics in the framework of a "contentual" or internal logic of arithmetic.

Keywords: Intuitionistic disjunction and existential quantifier, numerical existence, numerical content, Kronecker's general arithmetic, modular polynomial logic.

Introduction

Intuitionistic logic and intuitionistic number theory have the disjunction property and the numerical existence property. The question is to what extent these properties imply a notion of numerical content. The objective of this paper is to evaluate the claims about the realizability conditions of numerical existence and offer an alternative to intuitionistic logic and number theory in terms of a modular polynomial logic exhibiting a direct translation of logical formulas into an arithmetical logic internal to classical arithmetic. The term classical arithmetic is meant here to be contrasted to the set-theoretical Dedekind-Peano arithmetic formalized as Peano Arithmetic (PA). Classical arithmetic is designated as Fermat-Kronecker (F-K) arithmetic for classical number theory from Fermat to Gauss, and Kummer and Kronecker and beyond (see Gauthier [5]).

1 The disjunction and numerical existence properties.

The disjunction and numerical existence properties are easy to formulate. For a disjunction $A \vee B$ intuitionistic logic requires that one of the disjuncts be true or provable and for an existential quantifier $\exists x Ax$, it requires to exhibit a term t free for x in $A[x/t]$ denoting an instantiating element or object not otherwise specified in the BHK (Brouwer-Heyting-Kolmogorov) interpretation of intuitionistic logic. S.C. Kleene ([11], [12]) had the idea of calling such an object a *realizer*, an arbitrary witness or numerical instance in a given coding system. Kleene's realizability interpretation of intuitionistic logic adjoins a number n to a *realizer* such that the disjunction $A \vee B$ needs a pair (n, m) with values in $0, 1$ (for $0 = F$ and $1 = T$) with the proviso that if $n = 0$, then m realizes A and if $n = 1$, then m realizes B); for the existential quantifier, $\exists x Ax$ is realized by a pair (n, m), iff m is a *realizer* for $A(n)$.

The general setting of Kleene's realizability is the theory of partial recursive functions with recursive enumerability for which a partial recursive function is recursively realizable , iff some natural number n realizes it. This amounts to recursive enumerability for realized formulas in intuitionistic logic. For example, $\exists x Ax$ is proven, iff there is proof of Ax for some numeral x as in Gödel numbering. G. Kreisel has introduced a modified realizability interpretation, a typed variant with continuous functionals with the specific aim of reintroducing the notion of proof for a realized formula. H. Friedman ([3]) has shown in line with Kleene's work that realizability conditions allow to derive the numerical existence property in the set of axioms of a recursively enumerable extension T of Peano arithmetic where for the intuitionistic disjunction $A \vee B$, either A is a consequence of T or B is a consequence of T; for the existential quantifier, the numerical existence property stipulates that for each closed consequence $\exists x(Con(x))$ of T where x is a numerical variable, there is natural number n such that $Con(\tilde{n})$ is a consequence of T. All this is done within Peano arithmetic (extensions and fragments or substructures included) with the usual resources of recursive enumerability and set-theoretic machinery. However the realizability notion is not expressible in intuitionistic arithmetic HA^{ω} since it involves all recursive (partial) functions or functionals (of finite type). The situation is similar to the first number class in Cantorian set-theory for the sequence of natural numbers where the final segment $(0, \omega)$ is not expressible as an isomorphism type for its order type is incomparable or irreducible to any n in the ordinal polynomial of Cantor's normal form (see Gauthier [7]). All this means that the numerical existence property is not enough to produce numerical content, simply because logic is not arithmetic and that general computable functions do not generate feasible arithmetic or polynomial arithmetic the results of which can be computed in polynomial time. Here one should add that numerical witnesses are not arbitrary in a polynomial modular setting, since they are enumerated by a finite segment of an unlimited sequence of natural numbers in $\mathbf{N}$, of integers in $\mathbf{Z}$ or in finite fields $\mathbf{Q}$ as André Weil has taught us. As it is the case for modular arithmetic, Euclid's algorithm can act on modular logic for the elimination of logical constants and Fermat's infinite descent can be used to eliminate quantifiers in the translation of logic into arithmetic – e.g. by a calculus on binomial coefficients corresponding to a logical formula in propositional logic and decidable first-order monadic logic – . See Gauthier (5) and (6).

2 The notion of numerical content.

E. Bishop ([1]) has advocated the idea of mathematics as a numerical language. Here the author of the classic *Foundations of Constructive Analysis* deplores the fact that intuitionistic logic and mathematics are not constructive enough and a strict numerical interpretation of implication is needed simply because the usual

$$A \to B$$

amounts simply to the data of a proof of $A \to B$ effected by a construction which outputs a proof of A into a proof of B plus a proof of the said transformation wanting of any constructive information. It seems that Bishop was aiming at an existential instantiation for implication, but has been unable to provide with the right formulation and resorted finally to an appeal to Kronecker whom he considered closer to his foundational standpoint than was Brouwer. The proof-theorist U. Kohlenbach ([13]) claims that Gödel's functional interpretation of intuitionistic logic in Gödel ([8]) comes close to numerical content by the employment of primitive recursive functionals of finite type. Kohlenbach acknowledges though that the notion was already present in Hilbert's paper (Hilbert [10]), but he doesn't go back to Kronecker. I have shown that Hilbert was certainly inspired by Kronecker's own construction in (Kronecker [14]) and I have given the details of such a construction in (Gauthier [5], chap. 4).

3 Local negation.

Negation is interpreted "negatively" in intuitionistic logic as Bishop would say:

$$\neg A \equiv A \to 0 = 1(\text{absurdity})$$

and here he would lament the lack of numerical content. Gödel's interpretation in (Gödel [8]) comes to the same when he writes

$$\neg\, p \equiv\; p \supset 0 \cdot 1.$$

The *Dialectica* interpretation can have a direct polynomial interpretation (see Gauthier [5], chap. 7.9) and negation could be defined as 1- a on the pattern of relative complementation

$$a \to b = In\ ((X - a) \cup b)$$

for a topological space X, its Interior of open sets and b. Now, one can translate this in a combinatorial formula

$$a \to b = C\ ((2^n - a) + b)$$

where C stands for combinations of integer coefficients a, b of the polynomial $(a_0 x + b_0 x)^n$ with $a_0 x$ standing for $2^n - a$ (2^n is here the finite arithmetical universe as the power set of n integers). See below section 5 for more details on this construction.

The minus sign also appears in Y. Gurevich's treatment (Gurevich [9]) of Nelson's constructible falsity (Nelson [15]) which is expressed in terms of Kleene's realizability notion

$$\neg A \supset 1 = 0.$$

For Gurevich's minus sign, one has

$$— (A \supset B) \equiv A \wedge -B$$

$$— \neg(A) \equiv A$$

$$— A \supset \neg A$$

and a deduction theorem stating

$$— A \supset A \supset B.$$

Local negation in (Gauthier [4]) could be seen as a still stronger notion, the minus sign in a congruence relation being arithmetical while Gurevich's strong negation is logical and set in a Kripke model for Nelson's notion of constructible falsity couched in Kleene's recursive realizability style. There again numerical content is only postulated under an apriori numerical existence property. I present in the following a scheme inspired by Kronecker's theory of forms, his divisor theory for homogeneous polynomials. Such a scheme is intended to procure a direct access to numerical content in an arithmetical (modular polynomial) logic as the internal logic of arithmetic.

4 Modular polynomial logic.

For the Kroneckerian background of modular polynomial, I summarize the polynomial translation of logical constants inspired by Kronecker's general arithmetic (*allgemeine Arithmetik*). See Kronecker (15) and Gauthier (6).

There are various ways to translate a formal system into the natural numbers, simple substitution of numerical variables as in Ackermann (1940), translation of logical into arithmetical operations as in Goodstein's equational calculus (1951). In view of our use of Kronecker's results, we choose the polynomial translation. We are going to need some facts about the ring of polynomials in one indeterminate in our consistency proof. We pass briefly over the preliminaries (the graded ring of two or more polynomials has the same convolution product, which is our main tool- a Grassmannian product could be used to the same effect).

Polynomials of the form

$$f = f_0 + f_1 x + f_2 x^2 + \ldots + f_n x^n$$

where the $f.$ are the coefficients with the indeterminate x build up the subring $K[x]$ of the ring $K[[x]]$ of formal power series. The degree of a polynomial is the degree of the last non-zero coefficient $k = n$, while the leading coefficient of a polynomial f of degree k is the constant f_k and f is called monic if its leading coefficient is 1. Thus polynomials are power series having only a finite number of non-zero coefficients. The involution or Cauchy product of two polynomials will play an important role in our translation; we write it

$$f \cdot g = \left(\sum_m f_m x^m\right)\left(\sum_n g_n x^n\right) = \left(\sum_m \sum_n f_m g_n x^{m+n}\right).$$

The sum $f + g$ of polynomials f and g is obtained by simply adding corresponding coefficients. Homogeneous polynomials have all their non-zero terms of the same degree and they can be put in the following convenient form

$$a_o x^m + a_1 x^{m-1} y + \ldots + a_m y.$$

We are interested in irreducible (= prime in $K[x]$) polynomials. Every linear polynomial is irreducible. $K[x]$ has the property of unique factorization and this fact will be crucial in our future developments[1].

[1]Kronecker had proven the unique factorization theorem in the following formulation: ≪ Every integral algebraic form(= polynomial) is representable as a product of irreducible (prime) forms in a unique way≫(see Kronecker 1882, p. 352). Kronecker is interested in the theory of divisibility for forms and considers primitive forms (forms with no common divisor greater than 1), rather than prime polynomials in his work. The notions of integral domain and unique factorization domain are direct descendants to that theorem.

4.1 The inner arithmetical model

When we write, for example,

$$\varphi_m(\exists x A x)[n+m+\ell \ldots] = 1, \textit{iff} \sum A_n \in D_m$$

we can drop the right part and write

$$\varphi_m(\exists x A x)[n+m+\ell \ldots] = < n+m+\ell \ldots > = 1$$

to mean that we have a complementary mapping (of the intuitionistic spread) $\xi : \mathbf{N} \to \mathbf{N}$, so that we really have a polynomial function which evaluates polynomials by sequences of natural numbers after having defined an evaluation map of formulas into polynomials. The whole process is made possible by substitution alone. Moreover, in category-theoretic language, the indeterminate x is a universal element for the functor $U(\varphi(x)) = n$ for an integer n. If we look at variables of logical formulas as indeterminates, then any number of variables may be reduced to one.

We are going to make an essential use of Kronecker's notion of the content of forms in (1882, p. 343). A form M is contained in another form M' when the coefficients of the first are convoluted (combined in a Cauchy product) in the coefficients of the second. This idea of a content <*Enthalten-Sein*> of forms can be summarized in the phrase ≪The content of the product is the product of the contents (of each form)≫ which can be extracted from Kronecker's paper (1968, ll, 419-424). Thus, for a form to be contained or included in another form is simply to be linearly combined with it (to have its powers convoluted with the powers of the second form). We can adopt here a general principle of substitution - elimination formulated by Kronecker (1882). We state the *Substitution Principle*:

> 1) Two homogeneous forms (polynomials) F and F' are equivalent if they have the same coefficients (*i.e. content*);
>
> 2) Forms can be substituted for indeterminates (variables) provided the (linear) substitution is performed with integer coefficients.

We have immediately the following Proposition 1 (proposition X in Kronecker):

> Linear homogeneous forms that are equivalent can be transformed into one another through substitution with integer coefficients[2].

[2] This can be seen as the precursor of the problem of quantification over empty domains. We know that we have MP

$$\frac{A, A \supset B}{B}$$

in an empty domain, provided that A and B have the same free variables. But Kronecker had a more general theory of inclusion or content of forms in mind and the transformation in question is a composition of contents, an internal constitution of polynomials (forms) where indeterminates are not the usual functional variables.

We have also the following Proposition 2 (proposition X^0 in Kronecker):

> Two forms F and F' are absolutely equivalent, if they can be transformed into one another.

These propositions can be considered as lemmas for the unique factorization theorem for forms which Kronecker considered as one of his main results. The substitution procedure is simultaneously an elimination procedure, since indeterminates <*Unbestimmte*> are replaced by integer coefficients. Thus an indefinite (or effinite) supply of variables can be made available to a formal system and then reduced by the substitution-elimination method to an infinitely descending or finite sequence of natural numbers, as will be shown in the following. The equivalence principle makes it possible to have a direct translation between forms (polynomials) and (logical) formulas.

The substitution process takes place inside arithmetic, from within the Galois field F^*, i.e. the minimal, natural or ground field of polynomials which is the proper arena of the translation and indeterminates - Kronecker credits Gauss for the introduction of <*indeterminatae*> - are the appropriate tools for the mapping of formulas into the natural numbers. The important idea is that indeterminates in Kronecker's sense can be freely adjoined and discharged and although Kronecker did not always suppose that his forms were homogeneous, we restrict ourselves to homogeneous polynomials.

Definition : The height of a polynomial is the maximum of its lengths (number of its components or terms) -the height of a polynomial is indicated by a lower index. Let us rewrite the eight clauses of 2 in the polynomial fashion of the *valuation map* $\hat{\varphi}$.

Clause 1) An atomic formula A can be polynomially translated as

$$\hat{\varphi}(A)[n] = (a_o x)$$

(where the a_0 part is called the determinate and the x part the indeterminate and $\hat{\varphi}$ is the polynomial valuation function or map). Here the coefficient (a_o) corresponds to a given natural number (the "valuator") and 0 indicates that it is the first member of a sequence, x being its associate indeterminate. The polynomial $((a_o x))$ is thus a combination of the two polynomials (1,0,0,0 ...) and (0,1,0,0 ...). We identify polynomials by their first coefficients.

Clause 2) The negation of an atomic formula, that is $\neg A$, is translated as

$$\hat{\varphi}(\neg A)[n] = (1 - a_0 x)$$

Clause 3) The conjunction A and B is translated as $\hat{\varphi}(A \wedge B)(n \text{ x } m) = (a_0 x)\cdot(b_0 x)$ for the product of monomials $(a_0 x)$ and $(b_0 x)$.

Clause 4) The disjunction A or B is rendered by

$$(A \vee B)(n + m) = (a_0 x + b_0 x).$$

Clause 5) Local implication $A \to B$ is rendered by $\hat{\varphi}(A \to B)(m^n) = (\bar{a}_0 x + b_0 x)^n$ for $\bar{a}_0 x = 1 - a_0 x$.
Remarks : How is implication to be interpreted polynomially? A developed product of polynomials has the form

$$a \cdot b = \Big(\sum_i a_i x^i\Big)\Big(\sum_j b_j x^j\Big) = \Big(\sum_i \sum_j a_i b_j x^{i+j}\Big).$$

For a^b we could simply write $(a+b)^n$ for the binomial coefficients and put

$$(a_0 x + b_0 x)^n = a_o^x = na^{n+1} xbx + [n(n-1)/2!]a_2^{n-2} x^2 b_2 x^2 + \ldots + b_0^x x^n$$

in short

$$(a_0 x + b_0 x)^n_{i<n} = \sum_{i+j=n} (i+j) a^i b^j x^n.$$

The rationale for our translation is that we want to express the notion of inclusion of a in b by intertwining or combining their coefficients in a "crossed" product, the sum of which is 2^n which is also the sum of combinations of n different objects taken r at a time

$$\sum_{r=0}^{n} C_r^n.$$

Linear combination of coefficients is of course of central importance in Kronecker's view and one of his fundamental results is stated: ≪Any integral function of a variable can be represented as a product of linear factors≫ (1968, II, 209-247). In his (1968, III, 147-208), Kronecker refers to Gauss's concept of congruence and shows that a modular system with infinite (indeterminate) elements can be reduced to a system with finite elements. This is clearly the origin of Hilbert's basis theorem (1965, III, 199-257) on the finite number of forms in any system of forms with

$$F = A_1 F_1 + A_2 F_2 + \ldots + A_m F_m$$

for definite forms $F_1, F_2, \ldots, F_m$ of the system and arbitrary forms $A_1, A_2, \ldots, A_m$ with variables (indeterminates) belonging to a given field or domain of rationality <*Rationalitätsbereich*>. The fact that exponentiation is not commutative is indicated by the inclusion $a \subset b$. The combinatorial nature of implication is made more explicit in polynomial expansion and is strengthened by the symplectic (interlacing) features of local inclusion of content. We may also define implication, in analogy with the relative complement, as

$$(1^N - a_0 x) + b_0 x$$

where 1^N is the arithmetic universe polynomially expanded.
Clause 6) $\hat{\varphi}(\exists x Ax)[m+n+\ell\ldots] = \sum_0 (a_0x + b_0x + c_0x\ldots)_{i<n}$ where $\sum$ is an iterated sum of numerical instances with a_0 as the first member of the sequence.
Clause 7) $\hat{\varphi}(\forall x Ax)[n \times m \times \ldots \times \ell] = \prod_0 (a_0xb_0xc_0x)_{i<n}$.
Clause 8) $\hat{\varphi}(\mp x Ax)[n+m+\ell\ldots] = \prod_0 (a_0x + b_0x + c_0x\ldots)_n$ for the *effinite* quantifier.
Remarks : The *effinite* quantifier calls for some clarification. While the classical universal quantifier stands here for finite sets only, the *effinite* quantifier is meant to apply to infinitely proceeding sequences or *effinite* sequences. These are not sets and do not have a post-positional bound; we put an n to such sequences and a 2^n to sequences of such sequences

$$0, 1, 2, \ldots, n, \ldots, 2^n$$

with the understanding that n signifies an arbitrary bound. It should be pointed out that Boole in his *Mathematical Analysis of Logic* (1847) had also a universe (of classes) denoted by 1; negation was interpreted as $1 - x$. The fact that the ring $K[x]$ of polynomials enjoys the unique factorization property exhibited by infinite descent coupled with the proof by infinite descent of the infinity of primes makes essential use, from our point of view, of the *effinite* quantifier. We then have a combinatorial formulation

$$\prod_0^n (a_0xb_0xc_0x\ldots n_nx^n)$$

for the *effinite* quantifier; since $n! = \prod_{c<n} c$, the combinations of n. I call this scheme the absolute or standard scale. Any other scale is an associate scale (of indeterminates) and it is reducible by substitution to the standard scale.
As a foundational precept, there is no ω. Any transnatural or transarithmetic (transfinite, in Cantorian terminology) ordinal scale, *e.g.* up to ϵ_0 , is an associate scale and is by definition reducible. It is clear, from a Kroneckerian point of view, that Cantor's transfinite arithmetic becomes a dispensable associate (with an indeterminate pay-off!). The arithmetic universe n is naturally bounded by 2^n and not by $2^{\aleph_0}$ for infinite power series!

4.2 The consistency proof

Gentzen's pairing of reduction rules with transfinite inductions in the ϵ_0 segment may be looked at as an associate scale - the scale of ordinal numbers associated with every derivation (see Gentzen, 1969). The theorem of transfinite induction makes all ordinal numbers "accessible" by running through them in an increasing order; the reduction procedure then allows a descent according to the decreasing order of the ordinal numbers. In the same spirit, Takeuti attempts in (1975) a justification of transfinite induction by invoking the principle : $\ll$When all numbers smaller than β are recognized as accessible, the β is itself accessible$\gg$. But instead of strictly increasing sequences of ordinals $\beta_o < \beta_1 < \ldots < \beta_{\epsilon_0}$, Takeuti introduces directly strictly decreasing sequences $\mu > \ldots > \mu_1 > \mu_0$ for $\mu = lim(\omega^{\mu} n)$. As I have shown (see Gauthier, 1991), these ordinals are not uniformly recessible (over an immediate predecessor) and cannot count as ordinals in the absolute scale. On the other side, the associate scale can be reduced by a uniform procedure and can be entirely dispensed with, in accordance with Kronecker's general arithmetic.

Ackermann's consistency proof in (1940) also uses a decreasing sequence of ordinal indices in order to prove his finiteness result for global substitutions <*Gesamtersetzungen*> of fundamental types; his m-sequences are uniformly (immediately) recessible and the reduction procedure ends after a finite number of steps. However, despite the fact that his general recursion procedure is also built in the fashion of infinite descent, Ackermann must refer to the associate (indeterminate) scale of transfinite ordinals which he then reduces one-to-one to finite ordinals. But the transfinite ordinals are not immediately recessible and the upper bound estimate 2^{α} for indices of m-sequences (Ackermann, 1940, p. 193) has only a relative meaning, since it is not independent of some use of transfinite induction, as Ackermann admits[3]. Transfinite induction means always a detour via an infinite set.

Instead of the ordinal hierarchy of set-theoretic ascendency, I use here the arithmetic of irreducible polynomials to show the internal consistency of infinite descent in a direct way.

4.3 The elimination of logical constants

The connectives of negation, disjunction, conjunction are directly eliminable by translation into the arithmetic interpretation since they can be viewed as difference, sum and product of polynomials in a finite number of terms (constants and indeterminates or variables). We have then

Proposition 5.3.1 Connectives are eliminable through direct translation in the polynomial interpretation.

Proof. Rewrite the logical rules as follows for the sequent calculus with Γ

[3]Gödel's own consistency proof of arithmetic (The *Dialectica* interpretation) (1958) makes use of a general recursion schema (of functionals) over all finite types which is equivalent to complete induction. Herbrand's proof (1931) also requires general recursive functions. It is my contention that the concept of recursion stems from arithmetic reduction (recursion) procedures originating with Dedekind, but mainly from Kronecker's more algorithmic general arithmetic. Recursion is also "récurrence" which in France was another name for infinite descent.

the antecedent and Δ the (single) consequent, both consisting of polynomials (monomials); we write for negation

$$\frac{(\Gamma \cdot a_0x) \cdot \Delta}{\Gamma \cdot ((1 - a_0x) + \Delta)} \qquad \frac{\Gamma \cdot (a_0x + \Delta)}{(\Gamma \cdot ((1 - a_0x) \cdot \Delta}$$

with Δ empty *i.e.* "without content" in this case, or multiplication by zero and the understanding that the line has the meaning simply of an ordered sequence of sequents (consisting of sequences of formulas themselves). It should be obvious that we have replaced the sign $\vdash$ by the operation $\cdot$ in order to have polynomial uniformization which does not alter the meaning of the rules;
for disjunction :

$$\frac{\Gamma \cdot (a_0x + \Delta)}{\Gamma \cdot ((a_0xb_0x) + \Delta)} \qquad \frac{\Gamma \cdot (b_0x \cdot \Delta)}{\Gamma \cdot ((a_0xb_0x) + \Delta)}$$

and also

$$\frac{(\Gamma \cdot a_0x) \cdot \Delta \quad (\Gamma \cdot b_0x) \cdot \Delta}{(\Gamma \cdot (a_0x + b_0x)) \cdot \Delta}$$

for conjunction :

$$\frac{(\Gamma \cdot a_0x) \cdot \Delta}{(\Gamma \cdot (a_0x + b_0x)) \cdot \Delta} \qquad \frac{(\Gamma \cdot b_0x) \cdot \Delta}{(\Gamma \cdot (a_0x + b_0x)) \cdot \Delta}$$

and also

$$\frac{\Gamma \cdot (a_0x + \Delta) \quad \Gamma \cdot (b_0x + \Delta)}{\Gamma \cdot ((a_0x + b_0x) + \Delta)}$$

Remarks: We can treat implication as

$$\frac{\Gamma \cdot a_0x + b_0x + \Delta}{\Gamma \cdot ((a_0x) + b_0x) + \Delta} \qquad \frac{\Gamma \cdot (a_0x + \Delta_1) \quad (\Gamma \cdot b_0x) + \Delta_2}{\Gamma \cdot ((a_0x) \cdot b_0x) \cdot \Delta_1 + \Delta_2}$$

where Δ_1, and Δ_2 are two different sequences. There is some artificiality in the symmetrical treatment of intelim rules - the sagittal correspondence - in natural deduction systems (or in the sequent calculus). The symmetry induced by the inversion principle is not derived from the content (of symmetric polynomials), but from a formal duality which is not intrinsic or internal. Negation is generally not involutive- except in finite dual (Boolean) situations- and we could also introduce non-commuting variables in polynomials or in power series, while it is precluded by the double (dual) negation. In intuitionistic logic, this global symmetry is absent and the more complex situations that are reflected in the logic are an indication of more genetic, less structural features. Internal logic is an analysis of content. Here logical content = polynomial content. Finally, the detachment or elimination rule is equivalent to *Modus Ponens* and the polynomial translation should make manifest the content of the sequential character of inference. Gentzen's linear logic –Gentzen used the phrase "*lineares Räsonieren*"- is by itself a surface phenomenon of the polynomial content. The existential quantifier and the universal quantifier over finite sets interpreted as iterated (finite) sum and iterated (finite) product are also directly eliminable. We have

Proposition 5.3.2 The existential and universal quantifiers are eliminable through direct translation in the polynomial interpretation.

Proof. The universal quantifier can be rendered by

$$\frac{\Gamma \cdot (a_0 x + \Delta)}{\Gamma \cdot (\prod_i (a_i x^i) + \Delta)}(*) \qquad \frac{(\Gamma \cdot ax) \cdot \Delta}{(\Gamma \cdot (\prod_n (a_n x^n)) \cdot \Delta)}(**)$$

where (*) means that x is an indeterminate not appearing in Γ and (* *) means that x is an arbitrary term in the polynomial. The existential quantifier is translated as

$$\frac{\Gamma \cdot (ax + \Delta)}{\Gamma \cdot (\sum_n (a_n x^n) + \Delta)}(**) \qquad \frac{(\Gamma \cdot ax) \cdot \Delta}{(\Gamma \cdot (\sum_i (a_i x^i)) \cdot \Delta}(*)$$

Remarks: The terms $a_i x^i$ are arbitrary. Since we deal with polynomials (with integer coefficients), the existence property for the existential quantifier is immediately garanteed and since the (classical) universal quantifier is limited to finite domains, its scope is always well-defined.

4.4 The elimination of implication

We want to arithmetize (local) implication. We put $1-a = \bar{a}$ for local negation. We have $(\bar{a}_o x + b_o x)^n)$ and we want to exhaust the content of implication — in Gentzenian terms, this would correspond to the exhibition of subformulas (the subformula property). We just expand the binomial by decreasing powers

$$(\bar{a}_o x + b_o x)^n) = \bar{a}_0^n x + n\bar{a}^{n-1} x b_0 x + [n(n-1)/2!]\bar{a}^{n-2} x b^2 x + \ldots + b_0^n x$$

where the companion indeterminate x shares the same power expansion. By an arithmetical calculation (on homogeneous polynomials that are symmetric *i.e.* with a symmetric function $f(x, y) = f(y, x)$ of the coefficients)

$$
\begin{aligned}
(\bar{a}_o x + b_o x)^n) &= \bar{a}_0^n x + \sum_{k=1}^{n-1} (n-1/k-1)\bar{a}_0^{k-1} x + (n-1/k) a_0^k x b_0^{n-k} x + b_o^n x \\
&= \sum_{k=1}^{n} (n/k-1) a_0^k x b_0^{n-k} x + \sum_{k=0}^{n-1} (n-1/k) a_0^k x b_0^{n-k} x \\
&= \sum_{k=0}^{n-1} (n-1/k) a_0^{k+1} x b_0^{n-k} x + \sum_{k=0}^{n-1} (n-1/k) a_0^k x b_0^{n-k} x \\
&= \bar{a}_0 \sum_{k=0}^{n-1} (n-1/k)(\bar{a}_0 - 1)^k b^{n-1-k} x + \sum_{k=0}^{n-1} (n-1/k)\bar{a}_0^k x (b_0 - 1)^{n-1-k} x \\
&= (\bar{a}_1 x + b_1 x)(a_1 x + b_1 x - 1)^{n-1}
\end{aligned}
$$

and continuing by descent and omitting the x's, we have

$$
\begin{gathered}
(\bar{a}_2 + b_2)(\bar{a}_2 + b_2 - 2)^{n-2} \\
\cdots \quad \cdots \quad \cdots \quad \cdots \\
(\bar{a}_{n-2} + b_{n-2} + \bar{a}_{n-2} +_{n-2} -(n-2))^{(n-(n-2))} \\
(\bar{a}_{n-1} + b_{n-1} + \bar{a}_{n-1} +_{n-1} -(n-1))^{(n-(n-1))} \\
(\bar{a}_n + b_n)(\bar{a}_n + b_n)^{n-n}.
\end{gathered}
$$

Applying descent again on $(\bar{a}_n + b_n)$,we obtain

$$(\bar{a}_0 + b_0)$$

or, reinstating the x's

$$(\bar{a}_0 x + b_0 x).$$

Remembering that

$$(\bar{a}_x + b_x)^n_{k<n} = \sum_{k+m=n} (k+m/k)\bar{a}^k b^m x^n$$

we have

$$(\bar{a}_x + b_x)^{n+m=n}_{k<n} = \prod_{k+m=n} (k,m) = 2^n$$

or more explicitly

$$\sum_{i=0}^{m+n} c_1 x^{m+n=1} = \bar{a}_0 x \cdot b_0 x \prod_{i=1}^{m+n} (1 + c_i x) = 2^n$$

where the product is over the coefficients (with indeterminates) of convolution of the two polynomials (monomials) a_0 and b_0. We could of course calculate the generalized formula for polynomials

$$(a_0 x + b_0 x + c_0 x + \ldots + k_0 x)^n = \sum_{p,q,r\ldots s} a^p b^q c^r \ldots k^s$$

in the same manner, but we shall postpone the general case till we come to the effinite quantifier for a unified treatment.

The combinatorial content of the polynomial is expressed by the power set 2^n of the n coefficients of the binomial. I contend that this combinatorial content expresses also the meaning of local (iterated) implication. Convolution exhibits the arithmetic connectedness that serves to render the logical relation of implication. Implication is seen here as a power of polynomials, a^k and b^m with $k < m$ having their powers summed up and expanded in the binomial expansion. Some other formula may be used for the product, but it is essential to the constructive interpretation that the arithmetic universe be bounded by 2^n. One way to make things concrete is to analyse $a \to b$ in terms of

$$a \to b = C((2^n - a) + b)$$

where C can stand for combinations or coefficients. The formula is an arithmetical analogue of the topological interpretation of intuitionistic implication.
Theorem 5.4.1 Local implication $a \to b$ can be eliminated by interpreting it as $(\bar{a} + b)^n$.
Proof. By the above construction.

Here I only want to show how is produced a direct polynomial *eliminative* translation of logical constants by rewriting intelim rules of Gentzen's natural deduction system into a polynomial language. The unique identity axiom becomes the equality axiom $A = A$. There are also intelim rules and a polynomial translation for the *effinite* quantifier $\mp x Ax$ as a quantification over an unlimited sequence of natural numbers.

$$(\mathrm{I} \wedge) \quad \frac{A \quad B}{A \wedge B} \qquad ; \; a_0x, b_0x \equiv a_0x \cdot b_0x$$

$$(\mathrm{E} \wedge) \quad \frac{A \wedge B}{A} \quad \text{and} \quad \frac{A \wedge B}{B} \qquad ; \; a_0x \cdot b_0x \equiv a_0x, b_0x$$

$$(\mathrm{I} \vee) \quad \frac{A}{A \vee B} \quad \frac{B}{A \vee B} \qquad ; \; a_0x + b_0x \equiv a_0x, \; a_0x + b_0x \equiv b_0x$$

$$(\mathrm{E}\vee)\quad \frac{A\vee B \qquad \begin{array}{c}[A]\\ \vdots \\ C\end{array} \qquad \begin{array}{c}[B]\\ \vdots \\ C\end{array}}{C} \qquad ;\ \begin{array}{l} a_0x + b_0x \equiv c_0x \ (\mathrm{mod}\ b_0x) \\ a_0x + b_0x \equiv c_0x \ (\mathrm{mod}\ a_0x)\end{array}$$

$$(\mathrm{I}\to)\quad \frac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A\to B} \qquad ;\ a_0x \equiv b_0x (\mathrm{mod}\ a_0x + 1)$$

$$(\mathrm{E}\to)\quad \frac{A, A\to B}{B} \qquad ; 1 - a_0x \equiv b_0x (\mathrm{mod}\ a_0x)$$

$$(\mathrm{I}\neg)\quad \frac{\begin{array}{c}A\\ \bot\end{array}}{\neg A} \qquad ;\ 1 - a_0x \equiv 1 (\mathrm{mod}\ a_0x)$$

$$(\mathrm{E}\neg)\quad \frac{A, \neg A}{\bot} \qquad ;\ 1 - a_0x \equiv 0 (\mathrm{mod}\ a_0x)$$

$$(\mathrm{I}\forall)\quad \frac{Ax}{\forall x Ax} \qquad ;\ \prod_n a_0x^n \equiv a_0x (\mathrm{mod}\ n)$$

$$(\mathrm{E}\forall)\quad \frac{\forall x Ax}{At} \qquad ;\ a_0x \equiv \prod_n a_0x^n (\mathrm{mod}\ 1)$$

$$(\mathrm{I}\exists)\quad \frac{At}{\exists x Ax} \qquad ;\ \sum_n a_0x \equiv a_0x^n (\mathrm{mod}\ 1)$$

$$\text{(E } \exists) \quad \frac{\exists x A \qquad \begin{matrix}[Ax] \\ \vdots \\ B\end{matrix}}{B} \qquad ; \; a_0 x \equiv \sum_n b_0 x^n (\text{mod } 1)$$

$$\text{(I } \mp) \quad \frac{Ax_n}{\mp x A x} \qquad ; \; \prod_{n \ldots} a_0 x \equiv a_0 x^n (\text{mod } n \times n)$$

$$\text{(E } \mp) \quad \frac{\mp x A x}{A t_0} \qquad ; \; a_0 x \equiv \prod_{n \ldots} a_0 x^n (\text{mod } 1)$$

In translating logical formulas into congruent forms, we want to represent logical constants in a polynomial language in order to integrally arithmetize (polynomialize) logic. It is manifest in that context that deduction expressed in a turnstile $A \vdash A$ or A/A is a congruence relation in a modular calculus. Implication is rewritten

$$(\bar{a}_o x + b_o x)^n$$

for $\bar{a}_o x = 1 - a_o x$, the local negation (complement) of logic; exponent n denotes the degree of the polynomial (content) of implication that we reduce in the following way by a calculus on symmetrical polynomials (forms).
Remark. In structural and substructural logics, the deduction theorem

$$A, B \vdash C, \text{ iff } A \vdash B \to C$$

is also called *residuation* in the sense that A is a residue in

$$A + B = C, \text{ iff } A = C - B.$$

In those logics, the linear combinations of the premises are subjected to various complex rules to handle the residues. But in modular polynomial logic, the residue A is associated to a positive integer multiple n (An) via a congruence relation

$$C \equiv B (\text{mod } n)$$

meaning that $C - B$ is divisible by n, thus adding a direct numerical content to the notion of residuation. In the first three cases above (I $\wedge$), (E $\wedge$) and (I $\vee$), we could have added (mod 0) showing that the congruence relation leaves no residue or remainder, that is

$$C \equiv B \;\; (\text{mod } 0) \text{ implies } C = B.$$

Our notion of congruence is arithmetical for modular polynomial arithmetic with integer coefficients in line with Gauss (who invented the concept) and Kronecker. The algebraic notion of congruence in structural algebraic logics does not subsume any numerical content.

5 Final remarks

As I mentioned earlier, it is the Fermat-Kronecker number theory, that is Kronecker's polynomial arithmetic with Fermat's infinite descent, which constitutes the foundational background of my work. Obviously, the foundational motive is alien to set-theoretical foundations and one could quote H.M. Edwards ([2] p. 97) on numerical extensions:

> It is usual in algebraic geometry to consider function fields over an *algebraically closed field* – the field of complex numbers or the field of algebraic numbers rather than over **Q** (the field of rational numbers). In the Kroneckerian approach, the transfinite construction of algebraically closed fields is avoided by the simple expedient of adjoining new algebraic numbers to **Q** as needed.

By transfinite construction, Edwards means clearly the use of set-theoretical devices like Zorn's lemma and model-theoretic tools like the ultrafilter lemma which are equivalent to the axiom of choice *de facto* absent of Kronecker's general arithmetic (*allgemeine Arithmetik*) of polynomials. Algebraic extensions cannot be constructively defined in general, except in finite fields with explicit numerical extensions. For example, infinite models of set theory have elementary (first-order) extensions, e.g. generic sets of Cohen's forcing relation (including its Boolean-valued models) which by the way mimicks the method of field extensions, the accessibility relation on possible worlds in a Kripke model mimicking in turn a timelike forcing relation. Such set-theoretical and logical techniques do not have any potential for concrete numerical content and could be defined as transcendental constructions over infinite sets from a Kroneckerian point of view. So-called constructive or intuitionistic type theories (as in Martin-Löf's proposals) claim to do without the excluded middle principle and the axiom of choice in the construction of types, but as soon as the finite type territory is trespassed with transfinite induction (and recursion), excluded middle is reintroduced — as noticed by Kolmogorov already in 1925 (see Gauthier [5], chap. 6.4) — together with some version of the axiom of choice (e.g. dependent choice). One could add that Peano arithmetic, Heyting arithmetic with transfinite induction and their subtheories or extensions, such as Gödel's *Dialectica* interpretation with induction on all finite types, could not be made to have direct access to numerical content and numerical extensions in virtue of their lack in concrete constructive procedures and elementary arithmetical operations. The moral of this story may be drawn from Edward Nelson's *Predicative Arithmetic* (p. 177) in his program of arithmetization of logic. Nelson argues that impredicative arithmetic uses induction and recursion principles which need witnesses of witnesses of witnesses...for proofs of consistency, e.g. Gentzen's proof with reduction steps coupled by numerals associated with transfinite ordinals or realizability theories necessitating multiple numerical witnesses for the same logical formula. The proposal in this paper is a direct translation of logical constants into modular polynomial arithmetic with infinite descent replacing an induction postulate. Fermat's (truly finite) descent needs only finite natural numbers as direct witnesses as they are the only testifiers or verifiers of the arithmetical process. My own project for an arithmetical logic dates back to my paper in 1989 " Finite Arithmetic with Infinite Descent" *Dialectica*, 43(4): 329-337. I had sent a preprint to the great French arithmetician André Weil who had inspired my work. He responded that he approved of my use of infinite descent, but he didn't want to comment on my attempted formalization of infinite descent saying that he was not enough of a logician "trop peu logicien" (letter from André Weil, dated March 23, 1988 from Princeton Institute for Advanced Study).

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An Algebraic Approach to Orthologic

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Abstract

The literature about quantum theories emphasizes that the algebraic structures associated to orthodox quantum mechanics are non-distributive. In this paper we present a usual development on quantum algebras, the ortholattices, and a correspondent deductive system associated to them, the orthologic. Then, we show the adequacy between the algebraic ortholattices and the propositional orthologic using specifically algebraic models.

Keywords: Ortholattice, orthologic, algebraic model.

Introduction

The algebras of quantum theories are non-distributive and the corresponding logics are also non-distributive relative to the operators of conjunction and disjunction. Because of that it is not possible to use the famous theorem of Stones Isomorphism to establish their completeness. For more details on the beginning of quantum theories related to logic we suggest [1], [3], [6], [8] and [19].

Considering this non-distributivity originated from the non-distributivity of closed Hilbert spaces used in the foundation of Physical Theories, there is a tradition to associate to the quantum theories the basic algebraic structure named ortholattice as a first algebraic approximation.

Goldblatt [9] and Dalla Chiara, Giuntini and Greechie [6] have used an interesting semantic in Kripke's style to connect the algebraic models of ortholattices with the propositional quantum logic, the orthologic.

In this paper, we present algebraic aspects of quantum algebras and, then, we introduce a short deductive system very similar to those presented in above papers. We show some derivations on this Tarski system.

As an original contribution, we present a completely algebraic proof of soundness and completeness of orthologic relative to the ortholattices.

1 Algebras of quantum theories

Here we just present some elements of algebraic logic for the development of quantum logics. These elements are well known and can be met in several texts as [17], [18], [14], [5], [7] and [12].

Definition 1.1 *(Lattice) A lattice is an algebraic structure* $\mathbb{L} = \langle L, \curlywedge, \curlyvee \rangle$ *such that* L *is a non-empty set,* $\curlywedge$ *and* $\curlyvee$ *are two binary operations on* L *and for all* $a, b, c \in L$*:*

L_1 $(a \curlywedge b) \curlywedge c = a \curlywedge (b \curlywedge c)$ *and* $(a \curlyvee b) \curlyvee c = a \curlyvee (b \curlyvee c)$ *[associativity]*
L_2 $a \curlywedge b = b \curlywedge a$ *and* $a \curlyvee b = b \curlyvee a$ *[commutativity]*
L_3 $(a \curlywedge b) \curlyvee b = b$ *and* $(a \curlyvee b) \curlywedge b = b$ *[absorption].*

Proposition 1.2 *If* $\mathbb{L} = \langle L, \curlywedge, \curlyvee \rangle$ *is a lattice and* $a, b \in L$*, then it holds:*

L_4 $a \curlywedge a = a$ *and* $a \curlyvee a = a$ [idempotency]
L_5 $a \curlywedge b = a \Leftrightarrow a \curlyvee b = b$ [ordering].

Using condition L_5, we can define a relation of partial order on $\mathbb{L} = \langle L, \curlywedge, \curlyvee \rangle$.

Definition 1.3 *(Order)* $a \leq b \Leftrightarrow a \curlywedge b = a \Leftrightarrow a \curlyvee b = b$.

Proposition 1.4 *If* $\mathbb{L} = \langle L, \curlywedge, \curlyvee \rangle$ *is a lattice and* $a, b, c, d \in L$*, then:*

L_6 $a \leq a \curlyvee b$ *and* $b \leq a \curlyvee b$
L_7 $a \curlywedge b \leq a$ *and* $a \curlywedge b \leq b$
L_8 $a \leq c$ *and* $b \leq c \Rightarrow a \curlyvee b \leq c$
L_9 $c \leq a$ *and* $c \leq b \Rightarrow c \leq a \curlywedge b$
L_{10} $a \leq c$ *and* $b \leq d \Rightarrow a \curlyvee b \leq c \curlyvee d$
L_{11} $a \leq c$ *and* $b \leq d \Rightarrow a \curlywedge b \leq c \curlywedge d$.

We have defined lattice as an algebraic structure, but this concept can also be introduced as an ordering structure $\mathbb{L} = \langle L, \leq \rangle$.

Definition 1.5 *(Partial order) A binary relation* $\leq$ *on a non-empty set* L *is a partial order if the relation* $\leq$ *is reflexive, antisymmetric and transitive.*

Definition 1.6 *(Poset) A partially ordered set is a pair* $\langle L, \leq \rangle$ *such that* L *is a non-empty set and* $\leq$ *is a partial order on* L*.*

Definition 1.7 *(Supremum) Let* $\langle L, \leq \rangle$ *be a poset and* $a, b \in L$*. A supremum of* $\{a, b\}$*, if it exists, is an element* $c \in L$ *such that:*

(i) $a \leq c$ *and* $b \leq c$
(ii) *if* $a \leq d$ *and* $b \leq d$*, then* $c \leq d$.

A supremum, if it exists, is unique.

An infimum of $\{a, b\}$ is defined dually. It is unique, if it exists.

It is usual to denote the supremum of $\{a, b\}$ by $\sup\{a, b\}$ or $a \curlyvee b$ and the infimum of $\{a, b\}$ by $\inf\{a, b\}$ or $a \curlywedge b$. The supremum of $\{a, b\}$ is also named the *least upper bound* of $\{a, b\}$ and the infimum of $\{a, b\}$ is called the *greatest lower bound* of $\{a, b\}$.

If $\langle L, \leq \rangle$ is a poset such that for all $a, b \in L$ there exist $\inf\{a, b\}$ and $\sup\{a, b\}$, then the algebraic structure determined by $\langle L, \curlywedge, \curlyvee \rangle$ in which:

$$a \curlywedge b = \inf\{a, b\} \qquad \text{and} \qquad a \curlyvee b = \sup\{a, b\}$$

is a lattice.

It is straightforward to observe that these operations $\curlywedge$ and $\curlyvee$ satisfy the associative, commutative and absorption properties.

We can easily prove that the laws L_1 to L_{11} hold for the poset $\langle L, \leq \rangle$.

This way we can always see a lattice as a structure $\mathbb{L} = \langle L, \leq, \curlywedge, \curlyvee \rangle$.

Lemma 1.8 *If* $\mathbb{L} = \langle L, \leq, \curlywedge, \curlyvee \rangle$ *is a lattice, then:*

L_{12} $(a \curlywedge b) \curlyvee (a \curlywedge c) \leq a \curlywedge (b \vee c)$

L_{13} $a \curlyvee (b \curlywedge c) \leq (a \curlyvee b) \curlywedge (a \curlyvee c)$.

Proof. The result follows from L_6, L_7, and L_8. ■

Definition 1.9 *(Distributive lattice) A lattice* $\mathbb{L} = \langle L, \leq, \curlywedge, \curlyvee \rangle$ *is distributive if the following distributive laws are valid for all* $a, b, c \in L$*:*

L_{14} $(a \curlywedge b) \curlyvee c = (a \curlyvee c) \curlywedge (b \curlyvee c)$ *and* $(a \curlyvee b) \curlywedge c = (a \curlywedge c) \curlyvee (b \curlywedge c)$.

These are the right distributive laws and, due to the commutative property, the left distributive laws are also valid. Besides, only one of these two distributive laws would be enough to characterize the distributive property [14].

Definition 1.10 *(Lattices with* 0 *and* 1*) Let* $\mathbb{L} = \langle L, \leq, \curlywedge, \curlyvee \rangle$ *be a lattice. If* $\mathbb{L}$ *has the least element with respect to the order* $\leq$*, then this element is called the zero of* $\mathbb{L}$ *and is denoted by* 0*. On the other hand, if the lattice* $\mathbb{L}$ *has the greatest element with respect to the order* $\leq$*, then this element is called the one of* $\mathbb{L}$ *and it is denoted by* 1*.*

If the lattice $\mathbb{L}$ has the elements 0 and 1, then for every $a \in L$:

L_{15} $a \curlywedge 0 = 0$ and $a \curlyvee 0 = a$

L_{16} $a \curlywedge 1 = a$ and $a \curlyvee 1 = 1$.

We denote a lattice with 0 and 1 by $\mathbb{L} = \langle L, \leq, \curlywedge, \curlyvee, 0, 1\rangle$.

Definition 1.11 *(Pseudo-complement) Let $\mathbb{L} = \langle L, \leq, \curlywedge, \curlyvee, 0, 1\rangle$ be a lattice with 0 and 1. For $a \in L$, if there exists the element $-a = \max\{y \in L : a \curlywedge y = 0\}$ in L, then $-a$ is called the pseudo-complement of a.*

Definition 1.12 *(Pseudo-complemented lattice) A lattice $\mathbb{L}$ is called pseudo-complemented if every element $a \in L$ has a pseudo-complement $-a \in L$.*

Definition 1.13 *(Complement) Let $\mathbb{L} = \langle L, \leq, \curlywedge, \curlyvee, 0, 1\rangle$ be a lattice with 0 and 1. If $a \in L$, then an element $\overline{a} \in L$ is called a complement of a in $\mathbb{L}$ if:*
L_{17} $a \curlywedge \overline{a} = 0$
L_{18} $a \curlyvee \overline{a} = 1$.

The complement $\overline{a}$ is a pseudo-complement. But, for example, the intuitionistic pseudo-complement of Intuitionistic Logic is not a complement.

Definition 1.14 *(Complemented lattice) The lattice $\mathbb{L} = \langle L, \leq, \curlywedge, \curlyvee, 0, 1\rangle$ is called complemented if every element in L has a complement in L. If every element of L has exactly one complement, then the lattice $\mathbb{L}$ is called uniquely complemented.*

If the complement of a is unique, we will denote it by $\sim a$. If a lattice $\mathbb{L}$ is uniquely complemented, then we write $\mathbb{L} = \langle L, \leq, \sim, \curlywedge, \curlyvee, 0, 1\rangle$.

Lemma 1.15 *Let $\mathbb{L} = \langle L, \leq, \curlywedge, \curlyvee, 0, 1\rangle$ be a distributive lattice with 0 and 1. If there exists a complement of a, then it is unique.*
Proof. If y and z are two complements of a, then $a \curlywedge y = 0$, $a \curlyvee y = 1$, $a \curlywedge z = 0$, and $a \curlyvee z = 1$. As $z = 0 \curlyvee z = (a \curlywedge y) \curlyvee z = (a \curlyvee z) \curlywedge (y \curlyvee z) = 1 \curlywedge (y \curlyvee z) = y \curlyvee z$, we have, $y \leq z$. Analogously, $z \leq y$ and, hence, $z = y$. ■

Definition 1.16 *(Boolean algebra) A Boolean algebra $\mathcal{B}$ is a distributive and complemented lattice.*

The next results are particular cases of quantum algebras and good references are the texts [6] and [15].

Definition 1.17 *(Poset with involution) Let $\mathbb{L} = \langle L, \leq, 0, 1\rangle$ be a poset. An involution on $\mathbb{L}$ is a unary operation, denoted by $'$, such that for all $a, b \in L$:*
L_{19} $a = a''$
L_{20} $a \leq b \Rightarrow b' \leq a'$.

Then, $\mathbb{L} = \langle L, \ ', \leq, 0, 1\rangle$ is a poset with involution.

Proposition 1.18 *If* $\mathbb{L} = \langle L, \ ', \leq, 0, 1\rangle$ *is a poset with involution, then the De Morgan's laws hold:*

L_{21} $\quad (a \curlywedge b)' = a\,' \curlyvee b\,'$

L_{22} $\quad (a \curlyvee b)' = a\,' \curlywedge b\,'.$

Indeed, in view of L_{20} in $\mathbb{L} = \langle L, \leq, \ ', 0, 1\rangle$ the conditions L_{20}, L_{21} and L_{22} are equivalent.

Besides, in this case, $\sup\{a, b\}$ is defined if, and only if, $\inf\{a, b\}$ is also defined.

Definition 1.19 *(Ortholattice) An ortholattice is a complemented lattice with involution.*

We denote a such structure by $\mathbb{L} = \langle L, \leq, \ ', \curlywedge, \curlyvee, 0, 1\rangle$.

So in an ortholattice all the conditions L_1 - L_{22}, except distributivity L_{14}, are valid.

This way of including properties mirrors the achievement of Boolean algebra as in the tradition of Heyting algebras with intermediate algebras (Heyting algebra - Boolean algebra), with the difference of non-distributivity. The way from any ortholattice to a Boolean algebra has so many points and we can add several additional conditions or algebraic axioms depending on the path.

Following this context, the ortholattices are considered basic quantum structures.

In this paper we concentrate on ortholattices using only algebraic approach, which we shall posteriorly apply to the other quantum algebraic systems.

Like a last structure, let's define Kripke models as [6].

Definition 1.20 *(Kripke model) A model in the Kripke style for a language* $\boldsymbol{L}$ *has the following form:* $\mathcal{K} = (W, \vec{R}, \vec{o}, \mathcal{P}(W), v)$*, such that:*

(i) W *is a non-empty set of possible worlds;*

(ii) $\vec{R}$ *is a sequence of relations over* W*;*

(iii) $\vec{o}$ *is a sequence of operations defined over* W*;*

(iv) *the subsystem* $(W, \vec{R}, \vec{o})$ *is called the frame of* $\mathcal{K}$*;*

(v) $\mathcal{P}(W)$ *is the set of all subsets of* W*;*

(vi) $v : Var(\boldsymbol{L}) \to \mathcal{P}(W)$ *is a valuation that applies each variable into the set of all worlds where the variable is true or valid;*

(vii) *each valuation must preserve conditions that depend on the operators* $\vec{o}$ *of* $\boldsymbol{L}$*;*

(viii) *the valuations must be extended for the set of all formulas of* $\boldsymbol{L}$*.*

Usually we have only one binary relation R in the sequence $\vec{R}$, called the accessibility relation.

Considering that we almost always have relations involved in Kripke models, they are not exactly algebraic models, but a combination of algebraic and relational structures.

2 Logic of ortholattices

We present in this section the Orthologic, denoted by $\mathcal{OL}$, the logic of ortholattices, in a similar version to [9] and [6] and oriented by [20].

There is an interesting tradition on logic for quantum theories. We mention the following references: [10], [2], [11], [4] and [15].

The orthologic formalizes, in the logical language, some of essential characteristics of quantum theories that are unveiled by the orthoalgebras.

We do not have an algebraic conditional operator and circumvent this situation using a deductive system without any logical implication. We found this strategy for the first in [9].

The language of $\mathcal{OL}$ is indicated by $\mathbf{L}$.

The above literature shows aspects of quantum logics.

The propositional language $\mathbf{L}$ has exactly the operators $\neg$ for negation, and $\wedge$ for conjunction. Thus we take $\mathbf{L} = \{\neg, \wedge\}$.

The set of formulas of $\mathcal{OL}$ is denoted by $For(\mathbf{L})$ and the set of propositional variables by $Var(\mathbf{L}) = \{p_1, p_2, p_3, \ldots\}$. Of course $Var(\mathbf{L}) \subseteq For(\mathbf{L})$.

Thus, $For(\mathbf{L})$ is constructed from $Var(\mathbf{L})$ using only the symbols in $\mathbf{L} = \{\neg, \wedge\}$.

We do not have the disjunction $\vee$ as a basic operator in the language $\mathbf{L}$, but considering that in any ortholattice the De Morgan laws hold, we can define the disjunction of $\mathbf{L}$ by:

$$\varphi \vee \psi =_{df} \neg(\neg\varphi \wedge \neg\psi).$$

Definition 2.1 *(Configuration) For $\Sigma \cup \{\psi\} \subseteq For(\boldsymbol{L})$, a configuration is an expression of type $\Sigma \vdash \psi$.*

These configurations are schemes of formulas and we mean that we derive the consequence at right of $\vdash$ from the antecedent (a set of premises) at left of $\vdash$. The antecedent is a set of formulas and it is not required that it be a sequence or a finite multiset as in some calculus of sequents.

Derivation is a figure composed by a sequence of configurations.

For a formal definition we need to explicit the rules for derivations.

In general, if for $i \in \{1, 2, \ldots, n\}$, $\Sigma_i \cup \{\psi_i\} \subseteq For(\mathbf{L})$, then each rule has the form:

$$\frac{\Sigma_1 \vdash \psi_1, \ldots, \Sigma_{n-1} \vdash \psi_{n-1}}{\Sigma_n \vdash \psi_n},$$

with the meaning that from the premises, the configurations above the line, each rule permits the deduction of configuration $\Sigma_n \vdash \psi_n$.

The rules without premises are special cases, where the set of premises is empty, such that instead of: $\dfrac{\emptyset}{\Sigma \vdash \psi}$ we just write $\Sigma \vdash \psi$.
Of course, the configuration $\vdash \varphi$ must be understood as $\emptyset \vdash \varphi$.

Now, we present the properties of derivability for the logical system $\mathcal{OL}$.

This system does not have axioms, but only rules determined by the following configurations.

Rules without premises:

(ROL_1) $\{\varphi\} \vdash \varphi$ (auto-deductibility)

(ROL_2) $\{\varphi\} \vdash \neg\neg\varphi$ (double negation)

(ROL_3) $\{\neg\neg\varphi\} \vdash \varphi$ (double negation)

(ROL_4) $\{\varphi \wedge \psi\} \vdash \varphi$ (simplification)

(ROL_5) $\{\varphi \wedge \psi\} \vdash \psi$ (simplification)

(ROL_6) $\{\varphi \wedge \neg\varphi\} \vdash \sigma$ (explosion)

Rules with one premise:

(ROL_7) $\dfrac{\Gamma \vdash \varphi}{\Gamma \cup \Sigma \vdash \varphi}$ (monotonicity)

(ROL_8) $\dfrac{\{\psi\} \vdash \varphi}{\{\neg\varphi\} \vdash \neg\psi}$ (contraposition)

(ROL_9) $\dfrac{\{\varphi, \psi\} \vdash \sigma}{\{\varphi \wedge \psi\} \vdash \sigma}$ (left conjunction)

Rules with two premises:

(ROL_{10}) $$\frac{\Gamma \vdash \varphi,\ \Delta \cup \{\varphi\} \vdash \psi}{\Gamma \cup \Delta \vdash \psi} \text{ (cut)}$$

(ROL_{11}) $$\frac{\{\psi\} \vdash \varphi,\ \{\psi\} \vdash \neg\varphi}{\vdash \neg\psi} \text{ (absurdity)}$$

(ROL_{12}) $$\frac{\Gamma \vdash \varphi,\ \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi} \text{ (right conjunction).}$$

From auto-deductibility, monotonicity and cut, we observe that $\mathcal{OL}$ is a logic of Tarski. These three rules are considered structural rules, that is, rules without connectives. The other rules have the aim to put the particularities of an ortholattice in the propositional context.

Definition 2.2 *(Derivation) A derivation in $\mathcal{OL}$ is a finite sequence of configurations $\Sigma \vdash \psi$ such that each element in the sequence is a premise, or a rule without premises, or a conclusion of a rule whose premises are previous elements in the sequence.*

Definition 2.3 *(Derivable formula) A formula ψ is derivable from Σ if there is a derivation such that the last element of derivation is the configuration $\Sigma \vdash \psi$.*

Definition 2.4 *(Theorem) A formula ψ is a theorem of $\mathcal{OL}$ if it is derivable from the empty set, that is, $\emptyset \vdash \psi$ or $\vdash \psi$.*

Now we present some deduced rules in $\mathcal{OL}$.

(a) $\dfrac{\{\varphi\} \vdash \psi,\ \{\psi\} \vdash \sigma}{\{\varphi\} \vdash \sigma}$ (transitivity 1)

Consider the Cut $\dfrac{\Gamma \vdash \psi,\ \Delta \cup \{\psi\} \vdash \sigma}{\Gamma \cup \Delta \vdash \sigma}$ with $\Gamma = \{\varphi\}$ and $\Delta = \emptyset$.

(b) $\dfrac{\Gamma \vdash \psi,\ \{\psi\} \vdash \sigma}{\Gamma \vdash \sigma}$ (transitivity 2)

Consider the Cut $\dfrac{\Gamma \vdash \psi,\ \Delta \cup \{\psi\} \vdash \sigma}{\Gamma \cup \Delta \vdash \sigma}$ with $\Delta = \emptyset$.

(c) $\dfrac{\Gamma \vdash \psi,\ \Gamma \vdash \neg\psi}{\Gamma \vdash \varphi}$ (contradiction)

1. $\Gamma \vdash \psi$	premise
2. $\Gamma \vdash \neg\psi$	premise
3. $\Gamma \vdash \psi \wedge \neg\psi$	right conjunction in 1 and 2
4. $\{\psi \wedge \neg\psi\} \vdash \varphi$	explosion
5. $\Gamma \vdash \varphi$	(b) in 3 and 4.

(d) If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$

1. $\{\varphi\} \vdash \varphi$	auto-deductibility
2. $\Gamma \vdash \varphi$	monotonicity in 1.

(e) $\dfrac{\{\psi\} \vdash \varphi, \{\neg\psi\} \vdash \varphi}{\vdash \varphi}$ (proof by cases)

1. $\{\psi\} \vdash \varphi$	premise
2. $\{\neg\psi\} \vdash \varphi$	premise
3. $\{\neg\varphi\} \vdash \neg\psi$	contraposition in 1
4. $\{\neg\varphi\} \vdash \neg\neg\psi$	contraposition in 2
5. $\vdash \neg\neg\varphi$	absurdity in 3 and 4
6. $\vdash \varphi$	double negation in 5.

This system is particularly planned for derivations but not for proofs of theorems. However we can show some case of theorem.

(f) $\vdash \psi \vee \neg\psi$ (excluded middle)

1. $\{\varphi \wedge \neg\varphi\} \vdash \varphi$	simplification
2. $\{\varphi \wedge \neg\varphi\} \vdash \neg\varphi$	simplification
3. $\vdash \neg(\varphi \wedge \neg\varphi)$	absurdity in 1 and 2
4. $\vdash \neg\varphi \vee \neg\neg\varphi$	De Morgan in 3
5. $\vdash \psi \vee \neg\psi$	replacement in 4.

Goldblatt defined theorem in this logic as any formula φ such that $\psi \vee \neg\psi \vdash \varphi$ holds [9].

Proposition 2.5 $\{\varphi_1, \ldots, \varphi_n\} \vdash \psi \iff \varphi_1 \wedge \ldots \wedge \varphi_n \vdash \psi$.
Proof. ($\Rightarrow$) By $n-1$ applications of left conjunction.

($\Leftarrow$) By auto-deductibility we have $\{\varphi_i\} \vdash \varphi_i$, for $1 \leq i \leq n$. Then, by monotonicity $\{\varphi_1, \ldots, \varphi_n\} \vdash \varphi_i$, for $1 \leq i \leq n$. From that, applying right conjunction $n-1$ times we have $\{\varphi_1, \ldots, \varphi_n\} \vdash \varphi_1 \wedge \ldots \wedge \varphi_n$ and using the hypothesis and the transitivity 2 we have that $\{\varphi_1, \ldots, \varphi_n\} \vdash \psi$. ■

Proposition 2.6 *(Finite deductibility)* $\Sigma \vdash \psi \iff$ *there is* Σ_f *finite such that* $\Sigma_f \subseteq \Sigma$ *and* $\Sigma_f \vdash \psi$.
Proof. Each derivation is finite and uses only a finite number of formulas. ■

Corollary 2.7 $\Sigma \vdash \psi \iff$ *there are* $\varphi_1, \ldots, \varphi_n \in \Sigma$ *such that* $\varphi_1 \wedge \ldots \wedge \varphi_n \vdash \psi$.

Definition 2.8 *(Inconsistent and consistent sets) A set of formulas Σ is inconsistent if there is a formula ψ such that $\Sigma \vdash \psi \wedge \neg\psi$. The set Σ is consistent if it is not inconsistent.*

Definition 2.9 *(Deductive closure) The deductive closure of the set Σ is the set of all formulas derivable from Σ, that is, $\overline{\Sigma} = \{\varphi : \Sigma \vdash \varphi\}$.*

Of course $\Sigma \subseteq \overline{\Sigma}$.

Definition 2.10 *(Theory) Theory is a set of formulas deductively closed, that is, $\Sigma = \overline{\Sigma}$.*

3 Soundness

In this section we show that every derivation in $\mathcal{OL}$ is sound, that is, if we have a syntactical derivation $\Sigma \vdash \psi$, then we also have a consequence of ψ from Σ but in a semantic context.

As a first step we need to present this semantic consequence.

Definition 3.1 *(Restrict valuation) Let $\mathbb{L}$ be an ortholattice. A restrict valuation is a function $\breve{v} : Var(\boldsymbol{L}) \to \mathbb{L}$ that maps each variable of $\mathcal{OL}$ over an element of $\mathbb{L}$.*

Definition 3.2 *(Valuation) Valuation is a function $v : For(\boldsymbol{L}) \to \mathbb{L}$ that extends naturally and uniquely the function $\breve{v}$ as follows:*
(i) $v(p) = \breve{v}(p)$
(ii) $v(\neg\varphi) = v(\varphi)'$
(iii) $v(\varphi \wedge \psi) = v(\varphi) \curlywedge v(\psi)$.

Definition 3.3 *(Algebraic realization) Algebraic realization is a pair $(\mathbb{L}, v)$ such that $\mathbb{L}$ is an ortholattice and v is a valuation for $\mathcal{OL}$.*

Definition 3.4 *(Algebraic model) Let $\Gamma \subseteq For(\boldsymbol{L})$ and $(\mathbb{L}, v)$ an algebraic realization for $\mathcal{OL}$. Then $\mathcal{A} = (\mathbb{L}, v)$ is an algebraic model for Γ, or $\mathcal{A}$ satisfies Γ, if $v(\gamma) = 1$, for every $\gamma \in \Gamma$.*

We denote that $\mathcal{A} = (\mathbb{L}, v)$ is a model for Γ by $\mathcal{A} \vDash \Gamma$ and, in particular, if $\varphi \in For(\mathbf{L})$ and $v(\varphi) = 1$, then $\mathcal{A} \vDash \varphi$.

Definition 3.5 *(Validity in $\mathbb{L}$) A formula φ is valid in $\mathbb{L}$ if for every valuation v, the algebraic realization $\mathcal{A} = (\mathbb{L}, v)$ satisfies φ, that is, $\mathcal{A} \vDash \varphi$, for every valuation v.*

In this case we fix $\mathbb{L}$ but take any valuation v.

Definition 3.6 *(Valid formula) A formula φ is valid if it is valid in any algebraic realization $\mathcal{A}$.*

Now we do not fix any valuation v neither any ortholattice $\mathbb{L}$. We denote that φ is valid by $\vDash \varphi$.

We will denote any valid formula by $\top$, and any invalid formula by $\bot$. A formula is invalid if it is not valid in any algebraic realization.

Definition 3.7 *(Algebraic consequence relative to $\mathcal{A}$) Let $\Gamma \subseteq For(\boldsymbol{L})$ and $\mathcal{A} = (\mathbb{L}, v)$ an algebraic realization. A formula ψ is an algebraic consequence of Γ relative to $\mathcal{A}$, what is denoted by $\Gamma \vDash_{\mathcal{A}} \psi$, if:*

$$\textit{for any } b \in L, \textit{ if } b \leq v(\gamma) \textit{ for every } \gamma \in \Gamma, \textit{ then } b \leq v(\psi).$$

The idea is that $v(\psi)$ must be equal or bigger than the infimum of $\{v(\gamma) : \gamma \in \Gamma\}$. If $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$, then $\Gamma \vDash \psi \Leftrightarrow v(\gamma_1) \curlywedge \ldots \curlywedge v(\gamma_n) \leq v(\psi)$ and, in particular, $\{\gamma\} \vDash_{\mathcal{A}} \psi \iff v(\gamma) \leq v(\psi)$.

It is usual to define a similar consequence in the following way: [$*$] If $\Gamma \cup \{\psi\}$ is a set of formulas, then Γ implies ψ in the model $\mathcal{A}$, if $v_{\mathcal{A}}(\gamma) = 1$, for every $\gamma \in \Gamma$, then $v_{\mathcal{A}}(\psi) = 1$.

The above definition implies this condition [$*$], but they are not equivalent.

If we have some $\mathcal{A}$ in which $0 < v_{\mathcal{A}}(\psi) < v_{\mathcal{A}}(\gamma) < 1$, then, in accordance to [$*$] we have $\{\gamma\} \vDash \psi$, but it does not happen following the above definition of consequence.

The definition is perfect for the characterization of ortholattices.

Definition 3.8 *(Logical Consequence) A formula ψ is a logical consequence of Γ, or Γ implies ψ, what is denoted by $\Gamma \vDash \psi$, if for any algebraic realization $\mathcal{A}$, $\Gamma \vDash_{\mathcal{A}} \psi$.*

Now we can prove the Soundness Theorem.

Theorem 3.9 *If $\Gamma \subseteq For(\boldsymbol{L})$, then $\Gamma \vdash \gamma \Rightarrow \Gamma \vDash \gamma$.*

Proof. We need to show that each rule of $\mathcal{OL}$ preserves the validity.

Let $\mathcal{A} = (\mathbb{L}, v)$ be any algebraic realization. Then $\mathbb{L}$ is an ortholattice and each rule of $\mathcal{OL}$ is valid because:

(ROL_1): $v(\varphi) = v(\varphi)$.
(ROL_2) and ($0ROL_3$): $v(\varphi) = v(\neg\neg\varphi)$.
(ROL_4) and ($0ROL_5$): $v(\varphi \wedge \psi) = v(\varphi) \curlywedge v(\psi) \leq v(\varphi), v(\psi)$.

(ROL_6): $v(\varphi \wedge \neg\varphi) = v(\varphi) \curlywedge v(\varphi)' = 0 \leq v(\sigma)$, for any σ.

(ROL_7): $\Gamma \vDash \psi$, then for any $b \in L$, if $b \leq v(\gamma)$ for every $\gamma \in \Gamma$, then $b \leq v(\psi)$. Now, if we take $\Gamma \cup \Sigma$, the values of Σ do not invalidate the condition, because they may only vary with lower values. Hence $\Gamma \cup \Sigma \vDash \psi$.

(ROL_8): $\{\varphi\} \vDash \psi \Leftrightarrow v(\varphi) \leq v(\psi) \Leftrightarrow\ v(\psi)' \leq v(\varphi)' \Leftrightarrow \{\neg\psi\} \vDash \neg\varphi$.

(ROL_9): $\{\varphi, \psi\} \vDash \sigma \Leftrightarrow v(\varphi) \curlywedge v(\psi) \leq v(\sigma) \Leftrightarrow v(\varphi \wedge \psi) \leq v(\sigma) \Leftrightarrow \{\varphi \wedge \psi\} \vDash \sigma$.

(ROL_{10}): if $\Gamma \vDash \varphi$ and $\Sigma \cup \{\varphi\} \vDash \psi$, then then for any $b \in L$, if $b \leq v(\gamma)$ for every $\gamma \in \Gamma$ then $b \leq v(\varphi)$, and then for any $b \in L$, if $b \leq v(\sigma)$ for every $\sigma \in \Gamma \cup \{\varphi\}$ then $b \leq v(\psi)$. Thus, for any $b \in L$, if $b \leq v(\delta)$ for every $\delta \in \Gamma \cup \Sigma$ then $b \leq v(\psi)$, that is, $\Gamma \cup \Sigma \vDash \psi$.

(ROL_{11}): if $\{\psi\} \vDash \varphi$ and $\{\psi\} \vDash \neg\varphi$, then $v(\psi) \leq v(\varphi)$ and $v(\psi) \leq v(\neg\varphi)$, so $v(\psi) \leq v(\varphi) \curlywedge v(\varphi)' = 0$ and $v(\psi) = 0$. Thus $v(\neg\psi) = 1$ e hence $\vDash \neg\psi$.

(ROL_{12}): if $\Gamma \vDash \varphi$ and $\Gamma \vDash \psi$, then for any $b \in L$, if $b \leq v(\gamma)$ for every $\gamma \in \Gamma$ then $b \leq v(\varphi)$, and for any $b \in L$, if $b \leq v(\gamma)$ for every $\gamma \in \Gamma$, then $b \leq v(\psi)$. As $v(\varphi \wedge \psi) = v(\varphi) \curlywedge v(\psi)$, then for any $b \in L$, if $b \leq v(\gamma)$ for every $\gamma \in \Gamma$, then $b \leq v(\varphi \wedge \psi)$, that is, $\Gamma \vDash \varphi \wedge \psi$. ■

4 Completeness

Now we need to show that the set of logical consequences and derivable formulas are the same.

The proof of completeness for this logic was generally done using some Kripke model, as we can see for example in ([9], p. 26) and ([6], p. 181). Our proof below has a specifically algebraic character. Pavičić, [16] also presents an algebraic proof though less general than the following.

Definition 4.1 *(Full set) A set of formulas Δ is full if it is non-empty, consistent and holds:*

(i) *if $\varphi \in \Delta$ and $\{\varphi\} \vdash \psi$, then $\psi \in \Delta$;*

(ii) *if $\varphi, \psi \in \Delta$, then $\varphi \wedge \psi \in \Delta$.*

Proposition 4.2 *If Δ is full, then:*

(a) $\varphi, \psi \in \Delta \Leftrightarrow \varphi \wedge \psi \in \Delta$;

(b) $\Delta \vdash \varphi \Leftrightarrow \varphi \in \Delta$;

(c) $\top \in \Delta$.

Proof. *(a)* If $\varphi \wedge \psi \in \Delta$, as $\{\varphi \wedge \psi\} \vdash \varphi$ and $\{\varphi \wedge \psi\} \vdash \psi$, then by item (i) above $\varphi, \psi \in \Delta$.

(b) If $\Delta \vdash \varphi$, by Corollary 2.7, there are $\psi_1, \dots, \psi_n \in \Delta$ such that $\{\psi_1 \wedge \dots \wedge \psi_n\} \vdash \varphi$ and, by *(a)*, $\psi_1 \wedge \dots \wedge \psi_n \in \Delta$. So $\varphi \in \Delta$.

If $\varphi \in \Delta$, by example (d) $\Delta \vdash \varphi$.

(c) As for any Δ, $\Delta \vdash \top$, then, by *(b)*, $\top \in \Delta$. ■

From item (b), we observe that any full set is a theory.

Proposition 4.3 *If Δ_1 and Δ_2 are full, then $\Delta_1 \cap \Delta_2$ is full.*
Proof. If Δ_1 and Δ_2 are consistent, then $\Delta_1 \cap \Delta_2$ is consistent.

If $\varphi \in \Delta_1 \cap \Delta_2$ and $\{\varphi\} \vdash \psi$, then $\varphi \in \Delta_1$ and $\{\varphi\} \vdash \psi$, e $\varphi \in \Delta_2$ and $\{\varphi\} \vdash \psi$. As Δ_1 and Δ_2 are full, then $\psi \in \Delta_1$ and $\psi \in \Delta_2$. Hence $\psi \in \Delta_1 \cap \Delta_2$.

If $\varphi, \psi \in \Delta_1 \cap \Delta_2$, then $\varphi, \psi \in \Delta_1$ and $\varphi, \psi \in \Delta_2$. As Δ_1 and Δ_2 are full, then $\varphi \wedge \psi \in \Delta_1$ and $\varphi \wedge \psi \in \Delta_2$. Finally, $\varphi \wedge \psi \in \Delta_1 \cap \Delta_2$. ■

Proposition 4.4 $\Gamma \vdash \varphi \iff$ *φ belongs to every full extension of Γ.*
Proof. ($\Rightarrow$) Suppose that $\Gamma \vdash \varphi$ and Δ is a full extension of Γ. Then, by Corollary 2.7, there are $\varphi_1, \dots, \varphi_n \in \Gamma \subseteq \Delta$ such that $\{\varphi_1 \wedge \dots \wedge \varphi_n\} \vdash \varphi$. Moreover, by Definition 4.1 (ii) and (i), $\varphi_1 \wedge \dots \wedge \varphi_n \in \Delta$ and hence $\varphi \in \Delta$.

($\Leftarrow$) By contrapositive, suppose that Γ is consistent and $\Gamma \nvdash \varphi$.

Thus $\varphi \notin \overline{\Gamma}$ and, of course, $\Gamma \subseteq \overline{\Gamma}$. So, we show that $\overline{\Gamma}$ is full.

Since Γ is consistent, then $\overline{\Gamma}$ is consistent and $\overline{\Gamma} \neq \emptyset$.

Now:

(i) suppose $\psi \in \overline{\Gamma}$ and $\{\psi\} \vdash \delta$. Then $\Gamma \vdash \psi$ and $\{\psi\} \vdash \delta$ and so, by example (b), $\Gamma \vdash \delta$ and hence $\delta \in \overline{\Gamma}$.

(ii) if $\psi, \delta \in \overline{\Gamma}$, then there are $\psi_1 \wedge \dots \wedge \psi_n$, $\delta_1 \wedge \dots \wedge \delta_m \in \Gamma$ such that $\{\psi_1 \wedge \dots \wedge \psi_n\} \vdash \psi$ and $\{\delta_1 \wedge \dots \wedge \delta_m\} \vdash \delta$. Thus, by monotonicity and right conjunction, $\{\psi_1 \wedge \dots \wedge \psi_n \wedge \delta_1 \wedge \dots \wedge \delta_m\} \vdash \psi \wedge \delta$. Therefore, $\Gamma \vdash \psi \wedge \delta$ and $\psi \wedge \delta \in \overline{\Gamma}$.

Hence $\overline{\Gamma}$ is full. ■

Definition 4.5 *(Compatible sets) The sets Δ and Λ are compatible if there is no formula ψ such that $\Delta \vdash \psi$ and $\Lambda \vdash \neg\psi$.*

The next result is more properly an observation.

Proposition 4.6 *If Δ and Λ are compatible, then for every formula ψ, if $\Delta \vdash \psi$, then $\Lambda \nvdash \neg\psi$.*

Theorem 4.7 *If $\Delta \nvdash \neg\varphi$, then there exists Λ compatible with Δ such that $\Lambda \vdash \varphi$.*

Proof. Suppose that $\Delta \nvdash \neg\varphi$. If $\Lambda = \{\varphi\}$, by autodeductibility, $\Lambda \vdash \varphi$ and Δ and Λ are compatible, for on the contrary there is some formula σ such that $\Delta \vdash \neg\sigma$ and $\Lambda \vdash \sigma$. Then $\{\varphi\} \vdash \sigma$ and by contraposition $\{\neg\sigma\} \vdash \neg\varphi$. As $\Delta \vdash \neg\sigma$, by cut, $\Delta \vdash \neg\varphi$. ∎

Theorem 4.8 *Every consistent set Γ is included in a full set Λ.*

Proof. We take an enumeration $\psi_0, \psi_1, \psi_2, \ldots$ of $For(\mathbf{L})$ and construct a sequence of sets $\Lambda_i, i \in \mathbb{N}$, of compatible sets as in the previous theorem in the following way.

In the first step $\Lambda_0 = \Gamma$. So, if $\Lambda_n \vdash \neg\psi_n$, then $\Lambda_{n+1} = \Lambda_n \cup \{\neg\psi_n\}$ and if $\Lambda_n \nvdash \neg\psi_n$, then $\Lambda_{n+1} = \Lambda_n \cup \{\psi_n\}$. Thus, each set Λ_n is compatible with every previous set in the sequence and, by definition of compatible sets, they are consistent.

Finally, we take $\Lambda = \cup\Lambda_i, i \in \mathbb{N}$. This set is full, compatible with Γ and $\Gamma \subseteq \Lambda$. ∎

Now we must construct a canonical algebraic realization for $\mathcal{OL}$. Its domain is the set $\mathbf{T}$ of all full theories of $\mathcal{OL}$.

On $\mathbf{T}$ we need to determine a structure of an ortholattice.

Definition 4.9 *(Structure of full sets) For $\varphi, \psi \in For(\boldsymbol{L})$, we define $\widehat{\ }: For(\boldsymbol{L}) \to \mathcal{P}(\boldsymbol{T})$:*

(i) $\widehat{\varphi} = \{\Delta \in \boldsymbol{T} : \Delta \vdash \varphi\}$

(ii) $\widehat{\bot} = \emptyset$

(iii) $\widehat{\top} = \boldsymbol{T}$

(iv) $\widehat{\varphi} \curlywedge \widehat{\psi} = \widehat{\varphi} \cap \widehat{\psi}$

(v) $\widehat{\varphi}\,' = \{\Delta \in \boldsymbol{T} : \Delta$ *is incompatible with* $\widehat{\varphi}\}$.

Of course, for every $\Delta \in \mathbf{T}$, $\bot \notin \Delta$. On the other side, $\top$ belongs to all full sets. The conjunction coincides with set intersection, but the negation is not the set complement, because we would have a Boolean algebra with the classical negation. The classical complementation is a particular case of ortholattice complementation, however the quantum negation is weaker than the classical one.

Lemma 4.10 $\{\varphi\} \vdash \psi \Leftrightarrow \widehat{\varphi} \subseteq \widehat{\psi}$.

Proof. ($\Rightarrow$) If $\Delta \in \widehat{\varphi}$, then $\Delta \vdash \varphi$. Since $\{\varphi\} \vdash \psi$, then $\Delta \vdash \psi$ and hence $\Delta \in \widehat{\psi}$.

($\Leftarrow$) If $\{\varphi\} \nvdash \psi$, then there exists $\Delta \in \mathbf{T}$ such that $\Delta \vdash \varphi$ and $\Delta \nvdash \psi$. Thus, $\Delta \in \widehat{\varphi}$, but $\Delta \notin \widehat{\psi}$. So $\widehat{\varphi} \nsubseteq \widehat{\psi}$. ■

Now we need to prove the following important result.

Proposition 4.11 *The structure $\langle \boldsymbol{T}, \subseteq, {}', \curlywedge, \widehat{\bot}\rangle$ is an ortholattice.*
Proof. As the relation $\vdash$ is reflexive, transitive and antisymmetric, from the previous lemma, it follows that the relation $\subseteq$ is a partial order on $\mathbf{T}$. Besides, $\widehat{\bot}$ and $\widehat{\top}$ are the 0 and 1 on $\mathbf{T}$.

Then $\langle \mathbf{T}, \subseteq, {}', \curlywedge, \widehat{\bot}\rangle$ is a complemented partial order with 0 and 1, because: (i) $\widehat{\psi} \curlywedge \widehat{\neg\psi} = \widehat{\bot}$ and (ii) $\widehat{\psi} \curlyvee \widehat{\neg\psi} = \widehat{\top}$.

(i) $\widehat{\bot} = \emptyset \subseteq \widehat{\psi} \cap \widehat{\neg\psi} = \widehat{\psi} \curlywedge \widehat{\neg\psi}$. And $\Delta \in \widehat{\psi} \curlywedge \widehat{\neg\psi} \Rightarrow \Delta \in \widehat{\psi}$ and $\Delta \in \widehat{\neg\psi} \Rightarrow \Delta \vdash \psi$ and $\Delta \vdash \neg\psi \Rightarrow \Delta \vdash \bot$. So $\widehat{\psi} \curlywedge \widehat{\neg\psi} \subseteq \widehat{\bot}$.

(ii) $\widehat{\top} \subseteq \widehat{\psi}$ and $\widehat{\top} \subseteq \widehat{\neg\psi} \Rightarrow \widehat{\top} \subseteq \widehat{\psi} \curlyvee \widehat{\neg\psi} = \widehat{\psi} \curlyvee \widehat{\psi}\,'$. And $\Delta \in \widehat{\psi} \curlyvee \widehat{\neg\psi} \Rightarrow \Delta \in \widehat{\psi}$ or $\Delta \in \widehat{\neg\psi} \Rightarrow \Delta \vdash \psi$ or $\Delta \vdash \neg\psi \Rightarrow \Delta \vdash \psi \vee \neg\psi \Rightarrow \Delta \vdash \top \Leftrightarrow \Delta \in \widehat{\top}$. So $\widehat{\psi} \curlyvee \widehat{\neg\psi} \subseteq \widehat{\top}$.

Now we need to show that $'$ is an involution.

(iii) Suppose that $\widehat{\varphi} \neq \ \widehat{\varphi}\,''$. Thus either there is $\Delta_1 \in \mathbf{T}$ such that $\Delta_1 \vdash \neg\neg\varphi$ but $\Delta_1 \nvdash \varphi$, or there is $\Delta_2 \in \mathbf{T}$ such that $\Delta_2 \vdash \varphi$ but $\Delta_2 \nvdash \neg\neg\varphi$. We shall analyse only one case. As Δ_2 is full and $\{\varphi\} \vdash \neg\neg\varphi$, then $\Delta_2 \vdash \neg\neg\varphi$. In any case we have a contradiction.

(iv) By Lemma 5.10, $\widehat{\varphi} \subseteq \widehat{\psi} \Leftrightarrow \{\varphi\} \vdash \psi$ and, by Contraposition, $\{\varphi\} \vdash \psi \Leftrightarrow \{\neg\psi\} \vdash \neg\varphi$. Again by lemma $\widehat{\varphi} \subseteq \widehat{\psi} \Leftrightarrow \widehat{\neg\psi} \subseteq \widehat{\neg\varphi} \Leftrightarrow \widehat{\psi}\,' \subseteq \widehat{\varphi}\,'$. ■

Definition 4.12 *(Canonical valuation) A canonical valuation is any valuation $[.] : For(\boldsymbol{L}) \to \mathcal{P}(T)$ such that:*

(i) $[p] := \{\Delta \in \boldsymbol{T} : p \in \Delta\} = \widehat{p}$.

Proposition 4.13 *For every $\varphi \in For(\boldsymbol{L})$, it follows that $[\varphi] = \widehat{\varphi}$.*
Proof. By induction on the complexity of φ.

If φ is a propositional variable, then $[p] = \widehat{p}$, by the above definition.

If φ is of the type $\neg\psi$, then by induction hypotheses, $[\psi] = \widehat{\psi}$. So $[\varphi] = [\neg\psi] = [\psi]' = \widehat{\psi}\,' = \{\Delta \in \mathbf{T} : \Delta \textit{ is incompatible with } \widehat{\psi}\} = \{\Delta \in \mathbf{T} : \Delta \vdash \neg\psi\} = \widehat{\neg\psi} = \widehat{\varphi}$.

If φ is of the type $\psi \wedge \sigma$, then by induction hypotheses, $[\psi] = \widehat{\psi}$ and $[\sigma] = \widehat{\sigma}$. So $[\varphi] = [\psi \wedge \sigma] = [\psi] \curlywedge [\sigma] = \widehat{\psi} \cap \widehat{\sigma} = \widehat{\varphi}$. ■

Theorem 4.14 *(Strong completeness) If* $\Gamma \vDash \psi$*, then* $\Gamma \vdash \psi$.
Proof. If $\Gamma \nvdash \psi$, then $\Gamma \cup \{\neg\psi\}$ is consistent. By Theorem 4.8, there exists a full set Λ such that $\Gamma \cup \{\neg\psi\} \subseteq \Lambda$. As Λ is full and $\neg\psi \in \Lambda$, then $\Lambda \vdash \neg\psi$ and so $\Lambda \in \widehat{\neg\psi}$. Thus $\Lambda \vDash \neg\psi$. As Λ is full, then $\Lambda \nvDash \psi$ and considering that $\Gamma \subseteq \Lambda$, then $\Gamma \nvDash \psi$. ■

In this view the compactness is very simple.

Corollary 4.15 *(Compactness) If every finite* $\Gamma_f \subseteq \Gamma$ *has a model, then* Γ *has a model.*
Proof. If Γ does not have a model, then for every $\Delta \in \mathbf{T}$, it follows that $\Gamma \nsubseteq \Delta$. By Theorem 4.8, Γ is inconsistent. Hence, there is a formula ψ such that $\Gamma \vdash \psi$ and $\Gamma \vdash \neg\psi$, that is, there is a finite set $\Gamma_f \subseteq \Gamma$ such that $\Gamma_f \vdash \psi$ and $\Gamma_f \vdash \neg\psi$. Thus, the set Γ_f does not have a model. ■

5 Final remarks

We presented the ortholattices and a proof of adequacy between the algebraic ortholattices and the logic of ortholattices $\mathcal{OL}$ using only algebraic tools.

In the next steps we will try to include a conditional in $\mathcal{OL}$ and consider some specifications of ortholattices given by the introduction of new algebraic axioms. Of course, we must observe how the logical systems follow the algebraic inclusions.

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Truncation in Hahn Fields is Undecidable and Wild

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Abstract

We show that in any nontrivial Hahn field(a field of generalized power series) with truncation as a primitive operation we can interpret the monadic second-order logic of the additive monoid of natural numbers and the theory of such structure is undecidable. We also specify a definable binary relation on such a structure that has SOP(the strict order property) and TP_2(the tree property of the second kind).

Keywords: Hahn Fields, Truncation, undecidability, independence property, tree property of the second kind.

Introduction

Generalized series have been used in the past few decades in order to generalize classical asymptotic series expansions such as Laurent series and Puiseux series. Certain generalized series fields, such as the field of Logarithmic-Exponential Series [2], provide for a richer ambient structure, due to the fact that these series are closed under many common algebraic and analytic operations. In the context of generalized series the notion of truncation becomes an interesting subject of study. In the classical cases, a proper truncation of any Laurent series $\sum_{k\geq k_0} r_k x^k$ amounts to a polynomial in the variables x and x^{-1}. In the general setting a proper truncation of an infinite series can still be an infinite series. It has been shown by various authors [1, 3, 5] that truncation is a robust notion, in the sense that certain natural extensions of truncation closed sets and rings remain truncation closed. We here look at some first-order model theoretic properties of the theory of a Hahn Field equipped with truncation. We show that such theories are very wild in the sense that they can even interpret the theory of $(\mathbb{N};+,\times)$ via the interpretation of $(\mathbb{N},\mathcal{P}(\mathbb{N});+,\in)$, and are thus undecidable, solving a question posed by van den Dries. We also indicate definable binary relations with "bad" properties such as the strict order property and the tree property of the second kind. In section 1 we introduce the preliminaries of Hahn series and valued fields. In section 2 we recall the result of the undecidability of $(\mathbb{N}, \mathcal{P}(\mathbb{N}); +, \in)$ via the more familiar result of undecidability for $(\mathbb{N}; +, \times)$. The author would like to thank Philipp Hieronymi and Erik Walsberg for bringing the bibliography of monadic second-order logic to his attention.

Notations

We let m and n range over $\mathbb{N} = \{0, 1, \ldots\}$. For a set S, we denote its powerset by $\mathcal{P}(S)$. Given a set S, and a tuple of variables x we write S^x for the cartesian product $S^{|x|}$ where $|x|$ denotes the length of the tuple x. Given a language L, an L-formula $\phi(x)$, and an L-structure $\mathcal{M} = (M; \ldots)$ we let $\phi(M^x)$ denote the set $\{a \in M^x : \mathcal{M} \models \phi(a)\}$. For a field K we let $K^\times = K \setminus \{0\}$.

1 Preliminaries

Hahn Series

Let Γ be an additive ordered abelian group. Let $\mathbf{k}$ be a field. We indicate a function $f : \Gamma \to \mathbf{k}$ suggestively as a series $f = \sum_{\gamma \in \Gamma} f_\gamma t^\gamma$ where $f_\gamma = f(\gamma)$ and t is a symbol, and let $\operatorname{supp}(f) := \{\gamma \in \Gamma : f_\gamma \neq 0\}$ be the support of f. We denote the Hahn series field over $\mathbf{k}$ with value group Γ by

$$\mathbf{k}((t^\Gamma)) := \left\{ f = \sum_{\gamma \in \Gamma} f_\gamma t^\gamma : \operatorname{supp}(f) \text{ is well-ordered} \right\},$$

equipped with the usual operations of addition, and multiplication, that is with α, β, γ ranging over Γ:

$$f + g = \left(\sum_\gamma f_\gamma t^\gamma \right) + \left(\sum_\gamma g_\gamma t^\gamma \right) = \sum_\gamma (f_\gamma + g_\gamma) t^\gamma,$$

$$fg = \left(\sum_\gamma f_\gamma t^\gamma \right) \left(\sum_\gamma g_\gamma t^\gamma \right) = \sum_\gamma \left(\sum_{\alpha + \beta = \gamma} f_\alpha g_\beta \right) t^\gamma.$$

Let $f = \sum_\gamma f_\gamma t^\gamma$ be in $\mathbf{k}((t^\Gamma))$ and $\delta \in \Gamma$. The truncation of f at δ is $\sum_{\gamma < \delta} f_\gamma t^\gamma$ and we shall denote it by $f|_\delta$. We call f **purely infinite**, **bounded**, **infinitesimal** if $\operatorname{supp}(f) \subseteq \Gamma^{<0}$, $\operatorname{supp}(f) \subseteq \Gamma^{\geq 0}$, $\operatorname{supp}(f) \subseteq \Gamma^{>0}$, respectively. We will distinctly name three components of f: the **purely infinite part** $f|_0$, the **bounded part** $f_{\preccurlyeq} := f - f|_0$, and the **infinitesimal part** $f_{\prec} := f - f_{\preccurlyeq}$, so $f = f|_0 + f_{\preccurlyeq}, f_{\preccurlyeq} = f_0 + f_{\prec}$

2 Valued Fields

A **valued field** is a field K equipped with a surjective map $v : K \to \Gamma \cup \{\infty\}$, where Γ is an additive ordered abelian group, such that for all $f, g \in K$ we have

(V0) $v(f) = \infty \iff f = 0$,

(V1) $v(fg) = v(f) + v(g)$,

(V2) $v(f + g) \geq \min\{v(f), v(g)\}$

Every valued field gives rise to:

1. The valuation ring $\mathcal{O} := \{f \in K : v(f) \geq 0\}$, which is a local ring,
2. the maximal ideal $\mathcal{o} := \{f \in K : v(f) > 0\}$ of $\mathcal{O}$, and
3. the residue field $\mathbf{k} := \mathcal{O}/\mathcal{o}$ of K.

Example 2.1 *The canonical valuation on* $\mathbf{k}((t^\Gamma))$ *is given by the map* $v : \mathbf{k}((t^\Gamma))^\times \to \Gamma$ *where* $v(f) = \min(\operatorname{supp}(f))$. *We then observe that the corresponding valuation ring consists of the bounded elements of* $\mathbf{k}((t^\Gamma))$, *the maximal ideal of the valuation ring consists of the infinitesimal elements of* $\mathbf{k}((t^\Gamma))$ *and the residue field is isomorphic to* $\mathbf{k}$.

Other Structures in valued fields

A monomial group $\mathfrak{M}$ of a valued field K is a multiplicative subgroup of $K^\times$ such that for every $\gamma \in \Gamma$ there is a unique element $\mathfrak{m} \in \mathfrak{M}$ such that $v(\mathfrak{m}) = \gamma$.

Example 2.2 *Let* $K = \mathbf{k}((t^\Gamma))$. *Then the* ***canonical monomial group*** *of* K *is the set* $\{t^\gamma : \gamma \in \Gamma\}$.

An **additive complement to the valuation ring** $\mathcal{O}$ of a valued field K is an additive subgroup V of K such that $K = V \oplus \mathcal{O}$

Example 2.3 *Let* $K = \mathbf{k}((t^\Gamma))$. *Then the* ***canonical additive complement*** *for* K *is the set of purely infinite elements of* K.

3 The natural numbers

We start with the following well known result.

Theorem 3.1 *The theory of* $(\mathbb{N}; +, \times)$ *is undecidable.*

Monadic Second-Order Logic

Given a structure $\mathcal{M} = (M; \ldots)$, **monadic second-order logic of** $\mathcal{M}$ extends first-order logic over $\mathcal{M}$ by allowing quantification of subsets of M. More precisely it amounts to considering the two-sorted structure $(M, \mathcal{P}(M); \ldots, \in)$, where $\in \subseteq M \times \mathcal{P}(M)$ has the usual interpretation. The following Theorem and its proof appear in [4].

Theorem 3.2 *The theory of* $(\mathbb{N}, \mathcal{P}(\mathbb{N}); \in)$ *is decidable.*

Lemma 3.3 *Multiplication on* $\mathbb{N}$ *is definable in* $(\mathbb{N}, \mathcal{P}(\mathbb{N}); +, \in)$.

Proof. If the multiplication of consecutive numbers is defined, then general multiplication of two natural numbers can be defined in terms of addition:

$$n = mk \iff (m+k)(m+k+1) = m(m+1) + k(k+1) + n + n.$$

If divisibility is defined, then multiplication of consecutive numbers is defined by

$$n = m(m+1) \iff \forall k (\in \mathbb{N})(n|k \leftrightarrow [m|k \wedge (m+1)|k]).$$

Divisibility can be defined using addition by

$$m|n \iff \forall S(\in \mathcal{P}(\mathbb{N}))(0 \in S \wedge \forall x(\in \mathbb{N})(x \in S \rightarrow x + m \in S) \rightarrow n \in S).$$

Since addition is a primitive, multiplication is defined in $(\mathbb{N}, \mathcal{P}(\mathbb{N}); +, \in)$. ■

Corollary 3.4 *The theory of* $(\mathbb{N}, \mathcal{P}(\mathbb{N}); +, \in)$ *is undecidable.*

Proof. This follows from Lemma 3.3 and Theorem 3.1. ■

4 Hahn Fields with Truncation

Let $K = \mathbf{k}((t^\Gamma))$ be a Hahn field with non-trivial value group Γ. We consider K as an L-structure where $L = \{0, 1, +, \times, \mathcal{O}, \mathfrak{M}, V\}$, and the unary predicate symbols $\mathfrak{M}, \mathcal{O}$, and V are interpreted respectively as the canonical monomial group t^Γ, the valuation ring, and the canonical additive complement to $\mathcal{O}$. For $\gamma \in \Gamma$ and $\mathfrak{m} = t^\gamma$ we set $f|_\mathfrak{m} := f|_\gamma$. Then we have the equivalence (for f, $v \in K$)

$$f|_1 = v \iff v \in V \ \& \ \exists g \in \mathcal{O}(f = v + g),$$

showing that truncation at 1 is definable in the L-structure K. For $\mathfrak{m} \in t^\Gamma$ and $f \in K$ we have

$$f|_{\mathfrak{m}} = g \iff (\mathfrak{m}^{-1} f)|_0 = \mathfrak{m}^{-1} g,$$

showing that the operation $(f, \mathfrak{m}) \mapsto f|_{\mathfrak{m}} : K \times t^\Gamma \to K$ is definable in the L-structure K.

For convenience of notation we introduce the asymptotic relations $\preccurlyeq, \prec$, and $\asymp$ on K as follows. For $f, g \in K$, $f \preccurlyeq g$ if and only if there is $h \in \mathcal{O}$ such that $f = gh$, likewise $f \prec g$ if and only if $f \preccurlyeq g$ and $g \not\preccurlyeq f$, and $f \asymp g$ if and only if $f \preccurlyeq g$ and $g \preccurlyeq f$. Let $R := \{(\mathfrak{m}, f) \in t^\Gamma \times K : \mathfrak{m} \in t^{\operatorname{supp}(f)}\}$. Then R is definable in the L-structure K since for $a, b \in K$

$$(a, b) \in R \iff a \in t^\Gamma \text{ and } b - b|_a \asymp a.$$

Theorem 4.1 *The L-structure K interprets* $(\mathbb{N}, \mathcal{P}(\mathbb{N}); +, \in)$.

Proof. Let $\approx$ be the definable equivalence relation on K such that $f \approx g$, for $f, g \in K$, if and only if $\operatorname{supp}(f) = \operatorname{supp}(g)$. Take $\mathfrak{n} \in t^\Gamma$ such that $\mathfrak{n} \prec 1$. Consider the element $f = \sum_n \mathfrak{n}^n \in K$, and the set $S = \{g \in K : \operatorname{supp}(g) \subseteq \operatorname{supp}(f)\}$. Let $E \subseteq t^{\operatorname{supp}(f)} \times (S/{\approx})$ be given by

$$(\mathfrak{m}, g/{\approx}) \in E :\iff \mathfrak{m} \in t^{\operatorname{supp}(g)},$$

and note that E is definable in the L-structure K since R is. Define $\iota : \mathbb{N} \to t^{\operatorname{supp}(f)}$ by $\iota(n) = \mathfrak{m}^n$, and note that ι induces an isomorphism

$$(\mathbb{N}, \mathcal{P}(\mathbb{N}); \in) \xrightarrow{\sim} (t^{\operatorname{supp}(f)}, S/{\approx}; E),$$

such that $\iota(m + n) = \iota(m)\iota(n)$. ■

Corollary 4.2 *The theory of the L-structure K is undecidable.*

Proof. This follows easily from Theorem 4.1 and Theorem 3.4 ■

Defining the coefficient field k

We now consider $K = \mathbf{k}(t^\Gamma))$ as an L^--structure, where $L^- = \{0, 1, +, \times, \mathcal{O}, V\}$. Note that for $f \in \mathcal{O}$ we have

$$fV \subseteq V \iff f \in \mathbf{k},$$

where we identify $\mathbf{k}$ with $\mathbf{k}t^0$. Thus we can define the coefficient field $\mathbf{k}$ in the L^--structure K.

Question: Is it possible to define the monomial group t^Γ in the L'-structure K?

An approach without the monomial group

Alternatively we may work in the setting of the two sorted structure $(K, \Gamma; v, T)$ where K denotes the underlying field, Γ is the ordered value group, v is the valuation, and $T : K \times \Gamma \to K$ is such that $T(f, \gamma) = f|_\gamma$. Then we can define the binary relation $R \subseteq \Gamma \times K$ by

$$(\gamma, f) \in R :\Longleftrightarrow v(f - T(f, \gamma)) = \gamma.$$

We then obtain the following;

Theorem 4.3 *The two-sorted structure* $(K, \Gamma; v, T)$ *interprets* $(\mathbb{N}, \mathcal{P}(\mathbb{N}); +, \in)$.

The proof is similar to the proof of Theorem 4.1.

Corollary 4.4 *The theory of the two-sorted structure* $(K, \Gamma; v, T)$ *is undecidable.*

5 Dividing lines in model theoretic structures

We have already shown how $(\mathbb{N}; +, \times)$ can be interpreted in the L-structure K and thus we know that it has the strict order property and the tree property of the second kind among others. In this section we make explicit a binary relation that witnesses these properties inside K.

The independence property

Let L be a language and $\mathcal{M} = (M; \ldots)$ an L-structure. We say that an L-formula $\phi(x; y)$ **shatters** a set $A \subseteq M^x$ if for every subset S of A there is $b_S \in M^y$ such that for every $a \in A$ we have that $M \models \phi(a; b_S)$ if and only if $a \in S$. Let T be an L-theory. We say that $\phi(x; y)$ has the **independence property with respect to** T, or **IP** for short, if there is a model M of T, such that $\phi(x; y)$ shatters an infinite subset of M^x.

For a partitioned formula $\phi(x; y)$ we let $\phi^{opp}(y; x) = \phi(x; y)$, that is, ϕ^{opp} is the same formula ϕ but where the role of the parameter variables and type variables is exchanged.

Lemma 5.1 *A formula* $\phi(x; y)$ *has* IP *if* ϕ^{opp} *has* IP.

Proof. By compactness the formula $\phi(x; y)$ shatters some set $\{a_J : J \in \mathcal{P}(\mathbb{N})\}$. Let the shattering be witnessed by $\{b_I : I \subseteq \mathcal{P}(\mathbb{N})\}$. Let $B = \{b_{I_i} : i \in \mathbb{N}\}$ be such that $I_i = \{Y \subseteq \mathbb{N} : i \in Y\}$. Then we have

$$\models \phi(a_J, b_{I_i}) \iff i \in J,$$

and thus ϕ^{opp} shatters B. ∎

The Strict Order Property

We say that a formula $\phi(x; y)$ has the Strict Order Property, or SOP for short, if there are $b_i \in M^y$, for $i \in \mathbb{N}$, such that $\phi(M^x, b_i) \subset \phi(M^x, b_j)$ whenever $i < j$.

Proposition 5.2 *The formula $\varphi(x; y)$, defining the relation R as in section 4, has* SOP.

Proof. Let $\Theta = \{\theta_i : i \in \mathbb{N}\}$ be any subset of Γ such that $\theta_i < \theta_j$ for $i < j$, and consider the set $\{f_n = \sum_{i=0}^{n} t^{\theta_i} : i \in \mathbb{N}\}$. Note that $\varphi(K, f_m) \subset \varphi(K, f_n)$ for $m < n$. ■

The tree property of the second kind

We say that a formula $\phi(x; y)$ has the **tree property of the second kind**, or TP_2 for short, if there are tuples $b^i_j \in M^y$, for $i, j \in \mathbb{N}$, such that for any $\sigma : \mathbb{N} \to \mathbb{N}$ the set $\{\phi(x; b^i_{\sigma(i)}) : i \in \mathbb{N}\}$ is consistent and for any i and $j \neq k$ we have $\{\phi(x; b^i_j), \phi(x; b^i_k)\}$ is inconsistent.

Lemma 5.3 *If $\phi(x; y)$ has* TP_2 *then ϕ has* IP.

Proof. Let $\{\phi(x, b^i_j)\}_{i,j\in\mathbb{N}}$ witness TP_2 for $\phi(x; y)$. Fix j. Without loss of generality we will assume that $j = 0$. Consider the set $\{b^i_0\}$. Let $I \subseteq \mathbb{N}$. By TP_2 there is $a_I \in M^x$ such that

$$\mathcal{M} \models \phi(a_I; b^i_j) \iff (i \in I \text{ and } j = 0, \text{ or } i \notin I \text{ and } j = 1).$$

Thus by Lemma 5.1 $\phi(x; y)$ has IP. ■

Lemma 5.4 *Let $A = \{a_i : i \in \mathbb{N}\} \subseteq M^x$ and $B = \{b_I : I \in \mathcal{P}(\mathbb{N})\} \subseteq M^y$. Assume that there is $\phi(x; y)$ such that for any fixed $b_I \in B$*

$$\models \phi(a; b_I) \iff \text{there is } i \in I \text{ such that } a = a_i.$$

Then ϕ has TP_2.

Proof. Let ϕ, A, and B be as in the hypothesis of the Lemma. Let $P = \{p_i \in \mathbb{N}\}$ be the set of primes where $p_i \neq p_j$ for $i \neq j$. We construct $A^i_j \subseteq \mathbb{N}$ recursively as follows:

- $A^0_j := \{p_j^{n_0} : n_0 > 0\}$

- $A^i_j := \{p^m_{n_i} : n_i \in \mathbb{N},\ m \in A^{i-1}_j\}$.

So for example $A^1_2 = \{p_{n_1}^{p_2^{n_0}} : n_1, n_0 > 0\}$.
Claim 1 For $\alpha \in \mathbb{N}^n$ we have that $\bigcap_{i<n} A^i_{\alpha(i)} \neq \emptyset$.
It is not hard to check that

$$p_{\alpha(0)}^{\cdot^{\cdot^{\cdot^{p_{\alpha(n-1)}}}}} \in \bigcap_{i<n} A^i_{\alpha(i)}.$$

Claim 2: For fixed i, and $j \neq k$ we have $A^i_j \cap A^i_k = \emptyset$.
For simplicity in notation we prove the case where $i = 1$. Let $m \in A^1_j \cap A^1_k$. Then $m = p_{m_1}^{p_j^{m_0}} = p_{n_1}^{p_k^{n_0}}$. Since $p_j^{m_0}$ and $p_k^{n_0}$ are non-zero, we have that $p_{m_1} = p_{n_1}$, and thus $p_j^{m_0} = p_k^{n_0}$. Similarly, since m_0 and n_0 are non-zero we conclude that $p_j = p_k$, and thus $j = k$.

Now let $b^i_j = b_{A^i_j}$. By compactness, together with Claim 1, we get that the set $\{\phi(x; b^i_{\sigma(i)}) : i \in \mathbb{N}\}$ is consistent. By the hypothesis of the Lemma, together with claim 2, we get that for any i and $j \neq k$ we have $\{\phi(x; b^i_j), \phi(x; b^i_k)\}$ is inconsistent. ∎

If $\phi(x; y)$ and A are as in the lemma, we say that $\phi(x; y)$ and B **only shatter** A **in** M. Note that in this case A is in fact a definable set.

Proposition 5.5 *The formula $\varphi(x; y)$, defining the relation R as in section 4, has* TP$_2$.

Proof. Let Θ be a well-ordered subset of Γ and consider the sets

$$t^\Theta = \{t^\theta : \theta \in \Theta\}, \text{ and } B = \left\{\sum_{\delta \in \Delta} t^\delta : \Delta \subseteq \Theta\right\}.$$

It is clear then that $\varphi(x; y)$ and B only shatter t^Θ, and thus by Lemma 5.4 the formula $\varphi(x; y)$ has TP$_2$. ∎

Corollary 5.6 *The formula $\varphi(x; y)$, defining the relation R as in section 4, has* IP.

Proof. The result follows directly from proposition 5.5 and lemma 5.3. ∎

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A First-Order Modal Theodicy: God, Evil, and Religious Determinism

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Abstract

Edward Nieznański developed in 2007 and 2008 two different systems in formal logic which deal with the problem of evil [11, 12]. Particularly, his aim is to refute a version of the logical problem of evil associated with a form of religious determinism. In this paper, we revisit his first system to give a more suitable form to it, reformulating it in first-order modal logic. The new resulting system, called **N1**, has much of the original basic structure, and many axioms, definitions, and theorems still remain; however, some new results are obtained. If the conclusions attained are correct and true, then **N1** solves the problem of evil through the refutation of a version of religious determinism, showing that the attributes of God in Classical Theism, namely, those of omniscience, omnipotence, infallibility, and omnibenevolence, when adequately formalized, are consistent with the existence of evil in the world. We consider that **N1** is a good example of how formal systems can be applied in solving interesting philosophical issues, particularly in Philosophy of Religion and Analytic Theology, establishing bridges between such disciplines.

Keywords: Logical Problem of Evil, theodicy, formal theodicy, first-order modal logic, determinism, religious determinism.

Introduction

The problem of evil is one of the most famous issues in the history of philosophy. Among its formulations, David Hume's is one of the most famous. It states that the existence of God is in some sense incompatible with the existence of evil in the world. In his work *Dialogues concerning natural religion*, Hume makes the following considerations about God and His attributes:

> "Epicurus' old questions are yet unanswered. Is he willing to prevent evil, but not able? then is he impotent. Is he able, but not

> willing? then is he malevolent. Is he both able and willing? Whence then is evil?"[1]

Although scathing, Hume's allegation was not strong sufficiently to bring an effective trouble to theists' belief in God. As Alvin Plantinga affirms, it is not enough to put difficult questions to theism; the challenger should try to show that it is irrational to believe both in God and that evil exists in the world (cf. [13, p. 11]). In this direction, the philosopher John Mackie stated a version of the problem that now is known as the Logical Problem of Evil. Mackie says that theistic belief is positively *irrational*, for it is *contradictory*:

> "I think, however, that a more telling criticism can be made by way of the traditional problem of evil. Here it can be shown, not that religious beliefs lack rational support, but that they are positively irrational, that the several parts of the essential theological doctrine are *inconsistent* with one another." [9, p. 200.]

In other words, this challenge to theism is a problem of consistency between the existence of God and the existence of evil; it is a problem within the framework of classical logic. Many solutions were proposed to the Logical Problem of Evil, and among those, Plantinga's is the most famous. We will not explore his answer, and we do not put in question the merits of a Free Will Defense as a response to the problem. Anyway, even if it is correct, there are other correlate challenges to deal with. For instance, one could agree that the Free Will Defense is enough to deal with the questions raised by Mackie, but think that to have a theodicy, a stronger response to the Logical Problem of Evil, would be even better than a Defense as a philosophical response.[2]

Other questions regarding the Logical Problem of Evil are still relevant and can be more deeply explored. One of them is the question regarding the logical consistency between divine attributes like omnipotence and omniscience and the existence of evil. Regarding this, one can simply use the apparatus of Formal Logic, formalizing sentences like the attributes of God or the relation between God and the situations in the world, developing proofs and deducing rigorous results. These results, hopefully, can clarify the problem, solving ambiguities and exposing or demising contradictions through adequate tools provided by the vast field of Formal Logic.

Such a task is contemplated by the "Formal Theodicies" developed by the Polish philosopher and logician Edward Nieznański [11, 12]. He exposes in the

[1] [8, part 10, 23]

[2] Maybe Mackie had this in mind when recognized that Plantinga's solution dealt with his objections to theism, but still puts in question the character of his response as a "real solution" (cf. [1]).

abstract of his 2007 work his project of establishing a theodicy, that aims to show that the existence of God is logically compatible with the existence of evil:

> "The author of the article uses St. Thomas Aquinas' and G.W. Leibniz's philosophical inspirations to demonstrate by means of formal-logical means [sic] that inferring non-existence of evil from existence of God, as well as non-existence of God from existence of evil is a logical error." [11, p. 217.]

One way by which one can address the Problem of Evil is the following: it is contradictory to believe in the existence of the God of Classical Theism and in the existence of evil. God is omnipotent and omniscient, so one can assume that everything that happens is due to God's will. In other words, "if a situation is the case, then God wills such a situation to be the case", a determinist could argue.[3] This means that, if a situation is evil, then God wills such an evil situation, and this would contradict his omnibenevolence. To give an answer to such a determinism is the main concern of Nieznański in his papers, and is also ours here.

Let p denote a possible situation in the world, $P(p)$ denote "p is the case", and $\mathcal{C}_\theta$ denote "God wills", then, this claim can be formalized in the following way:

$$\textbf{(DET1)}\ \forall p(P(p) \to \mathcal{C}_\theta P(p))$$

(If a situation is the case, then God wills such a situation to be the case.)

Another determinist claim can be stated as follows: "if God knows that a situation is the case, then God wills such a situation to be the case". The relation between knowledge and will is more intricate in this claim; but perhaps, it is at least conceivable that if God knows a situation, but did not act in order to avoid it, it is because He willed it, for He is omnipotent. Thus, the proposition above requires an answer.

If $\mathcal{W}_\theta$ denotes "God knows", and the other symbols are interpreted as before, it is possible to formalize such a claim as follows:

$$\textbf{(DET2)}\ \forall p(\mathcal{W}_\theta P(p) \to \mathcal{C}_\theta P(p))$$

(If God knows that a situation is the case, then God wills such a situation to be the case.)

[3] By "situation" we mean a certain configuration of elements.

Originally, Nieznański developed two different approaches that aim at denying such religious determinism related to the Logical Problem of Evil. In order to do this, in both systems, Nieznański states the “constitutive properties of God”, in which the attributes of omniscience, infallibility, and omnipotence are formally described, as well as some other attitudes of God regarding situations. Then, he develops a formal axiology relating good, evil, and neutral situations to finally deal with versions of **DET1** and **DET2**.

However, although Nieznański's philosophical insights are penetrating and inventive, and his general methodology of formalization is very inspiring, some issues led us to revisit his first system [11] proposing some changes. Such changes can be summarized as follows: we proceed first by reestablishing the formal language to one that is logically more adequate according to our vision, defining it in first-order modal logic, as well as establishing the metalanguage, the rules of inference and other related features. Then, we define a new set of axioms schemes, many of them inspired in the work of Nieznański, but with a new formulation, to finally prove some theorems. Thus, the new resulting system, called **N1**, has much of the original basic structure, and many axioms, definitions and theorems still remain in a reformulate way; but, some new results are obtained. If the conclusions attained by **N1** are correct and true, then this system solves the problem of evil regarding religious determinism, showing that the attributes of God in Classical Theism, namely, those of omniscience, omnipotence, infallibility, and omnibenevolence, when adequately formalized, are logically compatible with the existence of evil in the world. We hope that it serves, as well, as a first presentation of some of Nieznański's insights published originally in Polish to a wider audience.

1 The system N1: language, rules, and axioms

The adaptation we make here from the system proposed by Nieznański is henceforward called **N1**. The basis of this system is a First-Order Modal Logic, i.e., a First-Order Classical Logic with the addition of two modal operators, $\mathcal{W}_\theta$ and $\mathcal{C}_\theta$.[4] The language $\mathcal{L}_{N1}$ of **N1** has the following symbols as primitives:

(i) Unary predicate symbols: B, P, d, z, n;
(ii) A binary predicate symbol: Op;
(iii) A symbol of constant (a distinguished element): θ;
(iv) Variables for situations: p, q, r, possibly with indexes;

[4] Among the works consulted are: the book written by Walter Carnielli and Claudio Pizzi about Modal Logics and Modalities [3], the widely-known introductory book of George Hughes and Max Cresswell on Modal Logic, specially chapter 13 [7], and Fitting and Mendelsohn's book on First-Order Modal Logic [5].

(v) The symbols for connectives: $\neg, \rightarrow$;
(vi) The symbol of universal operator: $\forall$;
(vii) Two symbols for specific modal operators: $\mathcal{C}_\theta, \mathcal{W}_\theta$.

The definition of a well-formed formula (abbreviated as *wff*) and the employ of parentheses is the usual, with the expected extensions. The formation rules are the following:

(FR1) Any sequence of symbols consisting of an n-ary predicate followed by n individual variables is a *wff*.
(FR2) If ϕ is a *wff*, so are $\neg\phi$, $\mathcal{W}_\theta\phi$, and $\mathcal{C}_\theta\phi$.
(FR3) If ϕ and ψ are *wff*, so is $(\phi \rightarrow \psi)$.
(FR4) If ϕ is a *wff* and v is a variable that stands for situations, then $\forall v\phi(v)$ is a *wff*.

Some rules of deduction of **N1** are: *Modus Ponens* (MP), *Uniform Substitution* (US), *Rule of Necessitation* (Nec) and *Substitution of Equivalents* (Eq). They are stated below:[5]

(MP) $\phi, \phi \rightarrow \psi \vdash_{N1} \psi$.
(US) [7, p. 25] The result of uniformly replacing any variable or variables $v_1, ..., v_n$ in a theorem by any *wff* $\phi_1, ..., \phi_n$, respectively, is itself a theorem.
(Nec) If $\vdash_{N1} \phi$, then $\vdash_{N1} \mathcal{W}_\theta\phi$ and $\vdash_{N1} \mathcal{C}_\theta\phi$.
(Eq) [7, p. 32] If ϕ is a theorem and ψ differs from ϕ in having some *wff* δ as a subformula at one or more places where ϕ has a *wff* γ as a subformula, then if $\gamma \leftrightarrow \delta$ is a theorem, ψ is also a theorem.

The Deduction Theorem **(DT)** is valid in the system:

Theorem 1 (Deduction Theorem). *If* $\phi, \Gamma \vdash_{N1} \psi$, *then* $\Gamma \vdash_{N1} \phi \rightarrow \psi$.[6]

Other symbols of the language are defined as follows (ϕ and ψ are *wff*s):

Def. 1. $\exists v\phi :\leftrightarrow \neg\forall v\neg\phi$

Def. 2. $(\phi \leftrightarrow \psi) :\leftrightarrow (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$

Def. 3. $(\phi \vee \psi) :\leftrightarrow (\neg\phi \rightarrow \psi)$

[5] Not all the rules presented here are used to prove the theorems of this paper, but by listing them we make explicit what is the modal characterization of our system.

[6] Hakli and Negri establish the conditions for using this theorem in modal logic: through defining a formal notion of derivation from assumptions, it is possible to prove the theorem for modal logics as stated above (cf. [6, p. 859-861]).

Def. 4. $(\phi \wedge \psi) :\leftrightarrow \neg(\phi \to \neg\psi)$

We will use the following convention:

$\alpha(p)$ stands for any *wff* that involves *only* the variable p, where p is free;

Thus, we will refer to a *wff* α that involves only a particular situation p as the 'state of affairs' $\alpha(p)$. We use the term 'state of affairs' here to indicate circumstances (possibly a fact) about a given situation.[7] So, any situation denoted by p is such that there are many states of affairs involving it. For instance, the formula $\alpha(p) \equiv P(p) \wedge \neg P(p)$ represents a state of affairs that does not occur, for it is contradictory.

For ease of reading, we have included in parentheses the standard interpretations for each *wff*. The following shall be considered as abbreviations or standard semantics in natural language:

$\theta :=$ 'God';
$P(p) :=$ 'p is the case';[8]
$B(\theta) :=$ 'θ is divine'.
$d(p) :=$ 'p is good';
$z(p) :=$ 'p is evil';
$n(p) :=$ 'p is neutral';
$K(p) :=$ 'p is contingent';
$Op(p, q) :=$ 'p is opposed to q';[9]
$\mathcal{C}_\theta\alpha(p) :=$ 'God wills the state of affairs $\alpha(p)$';
$\mathcal{W}_\theta\alpha(p) :=$ 'God knows the state of affairs $\alpha(p)$'.

As usual, all theorems, rules, and laws of Propositional Calculus are axioms, rules, and laws in our system, respectively. The abbreviation **PC** denotes steps in proofs that are based on rules and laws in Propositional Calculus; and the abbreviation **PC-Theorem** is used whenever a valid PC-schema is evoked.

[7] Naturally, those possible facts that are expressible in the language of **N1**.

[8] We do not need to assume here that 'to be the case', is the same as 'to be actual'. To say that 'p is the case' can be considered closer to 'p occurs' or to 'p has correspondence in reality' in a given considered world.

[9] We consider that two situations are opposite if they are contrary, that is, two opposite situations may at the same time both not be the case, but cannot at the same time both be the case. It will not be necessary here, but accordingly, we could assume as axioms to regulate Op the following formulas:

$$\forall p \forall q(Op(p, q) \leftrightarrow Op(q, p))$$
$$\forall p \forall q(Op(p, q) \to (P(p) \to \neg P(q))).$$

Thus, we present below the proper axioms of **N1**.[10]

The first axiom establishes that the distinguished element θ satisfies the primitive predicate B:

A1. $B(\theta)$

(God is divine.)

The second axiom corresponds to a quantification over the well known Axiom **4** of alethic modal logic, that characterizes the system **S4**, where the operator $\Box$ is substituted by $\mathcal{C}_\theta$.

A2. $\forall p(\mathcal{C}_\theta \alpha(p) \to \mathcal{C}_\theta \mathcal{C}_\theta \alpha(p))$

(For all situations, if God wills a state of affairs, then He wills to will such a state of affairs.)

The following axiom is, similarly, associated with the formula **5**, the characteristic axiom schema of **S5** system. One can see easily that there is an analogy between $\Diamond$, the operator of possibility in alethic modal logic, and $\mathcal{D}_\theta$, the operator of permission in **N1**:

A3. $\forall p(\mathcal{D}_\theta \alpha(p) \to \mathcal{C}_\theta \mathcal{D}_\theta \alpha(p))$

(For all situations, if God permits a state of affairs, then He wills to permit such a state of affairs.)

Another axiom here establishes something relevant, and easy to assume, in the context of the Logical Problem of Evil, i.e., that not all situations are good:

A4. $\neg \forall p(P(p) \to d(p))$

(Not all the situations that are the case are good.)

Next, we introduce three axioms in order to regulate the axiology of our system. The following axiom aims at capturing the attribute of omnibenevolence of God:

A5. $\forall p(\mathcal{C}_\theta P(p) \leftrightarrow d(p))$

(For all situations, God wills a situation to be the case *iff* the situation is good.)

The next axiom is one that relates good to evil situations:

[10]For simplicity, we will call simply axioms to both axioms properly speaking and axiom schemes.

A6. $\forall p(z(p) \rightarrow \neg d(p))$

(For all situations, if a situation is evil, then it is not good.)

In order to establish the relation between good and evil situations, when they are opposite, we assume the following axiom:

A7. $\forall p \forall q(Op(p, q) \rightarrow (d(p) \leftrightarrow z(q)))$

(For all situations, if two situations are opposite, then if one is good, the other is evil.)

The following axioms A8 and A9 introduce relations between will, opposition, and permission of God regarding states of affairs:

A8. $\forall p(\mathcal{S}_\theta \alpha(p) \leftrightarrow \mathcal{C}_\theta \neg \alpha(p))$

(For all situations, God opposes a state of affairs *iff* He wills the opposite.)

A9. $\forall p(\mathcal{D}_\theta \alpha(p) \leftrightarrow \neg \mathcal{S}_\theta \alpha(p))$

(For all situations, God permits a state of affairs *iff* He does not oppose to it.)

The next axiom states the relation between the opposition of God and neutral situations. Intuitively, a situation is neutral *iff* God does not will it and is not opposed to it. Therefore, if God is opposed to a situation, we can assume that such a situation is not neutral.

A10. $\forall p(\mathcal{S}_\theta P(p) \rightarrow \neg n(p))$

(For all situations, if God opposes a situation to be the case, then the situation is not neutral.)

Thus, each situation admits one of three possible axiological values. In this sense, the next axiom establishes that neutral situations are neither good nor evil.

A11. $\forall p(n(p) \leftrightarrow (\neg d(p) \wedge \neg z(p)))$

(For all situations, a situation is neutral *iff* it is neither good nor evil.)

The system **N1** has eleven axioms. The axiom A1 establishes that our distinguished element θ ('God') is divine. The axioms A2 and A3 govern the iteration and the composition of the operators $\mathcal{C}_\theta$ and $\mathcal{D}_\theta$, which clearly shows a modal character. The axiom A4 guarantees that there is at least one evil situation. The axiom A5 expresses that the will of God is the criterion for good. Axioms A6 and A7 provide a type of opposition between good and evil. Axioms A8 and A9 establish relations between the will and the opposition of God, and between the permission and the opposition of God with respect to states of affairs, while A10 establishes the relation between the opposition of God and neutral situations. Finally, the axiom A11 establishes that a situation is neutral iff such a situation is neither good nor evil.

In the following, we will give precise definitions of the divine attributes and deduce a series of theorems relevant to the solution of the Logical Problem of Evil and the constitution of a formal theodicy.

2 The attributes of God

The following definitions delineate some attributes of the God of Classical Theism and are essential to the understanding and discussion of the Logical Problem of Evil: omniscience, infallibility, and omnipotence.

Def. 5 (Omniscience of God). $WW :\leftrightarrow \forall p(\alpha(p) \to \mathcal{W}_\theta \alpha(p))$

(God is omniscient *iff* for all situations, if a state of affairs is the case, then God knows it.)

Def. 6 (Infallibility of God). $NM :\leftrightarrow \forall p(\mathcal{W}_\theta \alpha(p) \to \alpha(p))$

(God is infallible *iff*, for all situations, if God knows a state of affairs, then it is the case.)

Def. 7 (Omnipotence of God). $WM :\leftrightarrow \forall p(\mathcal{C}_\theta \alpha(p) \to \alpha(p))$

(God is omnipotent *iff*, for all situations, if God wills a state of affairs, then it is the case.)

Such definitions try to capture the historical conceptions and intuitions of the great religions that helped to shape an entire concept apparatus for the Classical Theism. It is not difficult to find foundations in such a religious and philosophical traditions to support such definitions, but we do not assign ourselves to such a task here.

The following definition sets what means to be 'divine' in the context of the system **N1**, according to our standard interpretation:

Def. 8 (God). $B(\theta) :\leftrightarrow WW \wedge NM \wedge WM$

(God is divine *iff* He is omniscient, infallible, and omnipotent.)

As God satisfies the predicate of divinity, we have theorem T1:

T1. $WW \wedge NM \wedge WM$

(God is omniscient, infallible, and omnipotent.)

Proof.
1. $B(\theta)$ [A1]
2. $B(\theta) :\leftrightarrow WW \wedge NM \wedge WM$ [Def. 8]
3. $WW \wedge NM \wedge WM$ [MP, 1, 2]

$\square$

Theorems T2, T3, and T4 are also easily deduced from T1, and describe extensively God's attitudes regarding states of affairs. As Nieznański observes about the corresponding theorems in his system, theorems T2 and T3 formalize a fact that is in agreement with the observation of Thomas Aquinas, who says that "God knows all things whatsoever that in any way are" [11, p. 204].[11]

T2. $\forall p(\alpha(p) \rightarrow \mathcal{W}_\theta \alpha(p))$ $\square$

(For all situations, if a state of affairs is the case, then God knows it.)

T3. $\forall p(\mathcal{W}_\theta \alpha(p) \rightarrow \alpha(p))$ $\square$

(For all situations, if God knows a state of affairs, then it is the case.)

Theorems T2 and T3 establish the correspondence between the knowing of God and the states of affairs that are the case. Every state of affairs that God knows is the case and, conversely, if a state of affairs is the case, then God knows it. Theorem T4, on the other hand, deals with another relevant attribute here: the omnipotence of God, related with what God wills:

T4. $\forall p(\mathcal{C}_\theta \alpha(p) \rightarrow \alpha(p))$ $\square$

(For all situations, if God wills a state of affairs, then it is the case.)

The following theorem states that God cannot will contradictions. This follows immediately from the underlying Classical Logic.

T5. $\neg \exists p(\mathcal{C}_\theta(\alpha(p) \wedge \neg \alpha(p)))$

(There is no situation such that God wills some contradiction.)

[11] "*Deus scit omnia quaecumque sunt quocumque modo*" (Thomas Aquinas, *Summa Theologiae*, I, q. 14, a. 9 co.).

Proof.
1. $\mathcal{C}_\theta(\alpha(p) \wedge \neg\alpha(p)) \to (\alpha(p) \wedge \neg\alpha(p)))$ [T4, $\alpha(p)/(\alpha(p) \wedge \neg\alpha(p))$, Spec]
2. $\neg(\alpha(p) \wedge \neg\alpha(p)) \to \neg\mathcal{C}_\theta(\alpha(p) \wedge \neg\alpha(p))$ [**PC**, 1]
3. $\neg(\alpha(p) \wedge \neg\alpha(p))$ [**PC**-Theorem]
4. $\neg\mathcal{C}_\theta(\alpha(p) \wedge \neg\alpha(p))$ [MP, 2, 3]
5. $\forall p \neg\mathcal{C}_\theta(\alpha(p) \wedge \neg\alpha(p))$ [Gen, 4]
6. $\neg\exists p \mathcal{C}_\theta(\alpha(p) \wedge \neg\alpha(p))$ [**PC**, 5]

□

It is easy to see that T5 could be derived using just the theorem stated in line 3 of the proof above and Necessitation Rule **(Nec)**. The same thing could be made with the operator $\mathcal{W}_\theta$: God cannot know contradictory states of affairs.

The following definition states what it means for God to be *coherent* regarding a situation, and T6 states another result about the will of God:

Def. 9 (Coherence)**.** $coherent_\theta(p) :\leftrightarrow (\mathcal{C}_\theta\alpha(p) \to \neg\mathcal{C}_\theta\neg\alpha(p))$

(God is said to be "coherent with Himself regarding a situation" whenever the following occurs: if He wills a state of affairs involving that situation, then He does not will the opposite.)

T6. $\forall p(\mathcal{C}_\theta\alpha(p) \to \neg\mathcal{C}_\theta\neg\alpha(p))$

(For all situations, if God wills a state of affairs, then it is not the case that He wills the opposite.)

Proof.
1. $\mathcal{C}_\theta\neg\alpha(p) \to \neg\alpha(p)$ [T4, $\alpha(p)/\neg\alpha(p)$, Spec]
2. $\neg\neg\alpha(p) \to \neg\mathcal{C}_\theta\neg\alpha(p)$ [**PC**, 1]
3. $\alpha(p) \to \neg\mathcal{C}_\theta\neg\alpha(p)$ [**PC**, 2]
4. $\mathcal{C}_\theta\alpha(p) \to \alpha(p)$ [T4, Spec]
5. $\mathcal{C}_\theta\alpha(p) \to \neg\mathcal{C}_\theta\neg\alpha(p)$ [**PC**, 4, 3]
6. $\forall p(\mathcal{C}_\theta\alpha(p) \to \neg\mathcal{C}_\theta\neg\alpha(p))$ [Gen, 4]

□

By definition, it follows from T6:

T7. $\forall p(coherent_\theta(p))$

(Regarding all situations, God is coherent with Himself.)

Proof.
1. $\mathcal{C}_\theta\alpha(p) \to \neg\mathcal{C}_\theta\neg\alpha(p)$ [T6, Spec.]
2. $coherent_\theta(p)$ [Def. 9, 1]
3. $\forall p(coherent_\theta(p))$ [Gen, 2]

□

Next, some theorems are stated in order to explore the relations between "attitudes" of God towards states of affairs.

T8. $\forall p(\mathcal{S}_\theta\alpha(p) \leftrightarrow \neg\mathcal{D}_\theta\alpha(p))$

(For all situations, God is opposed to a state of affairs *iff* He does not permits it.)

Proof.
1. $\mathcal{D}_\theta\alpha(p) \leftrightarrow \neg\mathcal{S}_\theta\alpha(p)$ [A9, Spec.]
2. $\neg\mathcal{D}_\theta\alpha(p) \leftrightarrow \mathcal{S}_\theta\alpha(p)$ [**PC**]
3. $\forall p(\mathcal{S}_\theta\alpha(p) \leftrightarrow \neg\mathcal{D}_\theta\alpha(p))$ [**PC**, Gen, 2]

□

The theorem below states the relation between $\mathcal{C}_\theta$ and $\mathcal{D}_\theta$:

T9. $\forall p(\mathcal{D}_\theta\alpha(p) \leftrightarrow \neg\mathcal{C}_\theta\neg\alpha(p))$

(For all situations, God permits a state of affairs *iff* He does not will the opposite.)

Proof.
1. $\mathcal{S}_\theta\alpha(p) \leftrightarrow \mathcal{C}_\theta\neg\alpha(p)$ [A8, Spec]
2. $\mathcal{S}_\theta\alpha(p) \leftrightarrow \neg\mathcal{D}_\theta\alpha(p)$ [T8, Spec]
3. $\mathcal{D}_\theta\alpha(p) \leftrightarrow \neg\mathcal{C}_\theta\neg\alpha(p)$ [**PC**, 1, 2]
4. $\forall p(\mathcal{D}_\theta\alpha(p) \leftrightarrow \neg\mathcal{C}_\theta\neg\alpha(p))$ [Gen, 3]

□

From T9 it is easy to recognize the analogy between alethic modal operators $\Box$ and $\Diamond$ and **N1** modal operators $\mathcal{C}_\theta$ and $\mathcal{D}_\theta$, respectively. It becomes also clear that T6 is linked to formula **D**, the axiom that characterizes the **KD** modal system; in terms of the equivalence stated in T9, T6 can be written as $\forall p(\mathcal{C}_\theta\alpha(p) \to \mathcal{D}_\theta\alpha(p))$.

It is easy to see that the following theorems can be deduced from T9:

T9.1 $\forall p(\neg\mathcal{D}_\theta\alpha(p) \leftrightarrow \mathcal{C}_\theta\neg\alpha(p))$ □

T9.2 $\forall p(\mathcal{D}_\theta\neg\alpha(p) \leftrightarrow \neg\mathcal{C}_\theta\alpha(p))$ □

T9.3 $\forall p(\neg\mathcal{D}_\theta\neg\alpha(p) \leftrightarrow \mathcal{C}_\theta\alpha(p))$ $\square$

One more analogy between **N1** and normal modal systems comes here: T10 below is related to the formula $\mathbf{T}^\Diamond$, valid in **KT** modal logic. Furthermore, it is philosophically meaningful, for it states the relation between the permission of God and the states of affairs:

T10. $\forall p(\alpha(p) \to \mathcal{D}_\theta\alpha(p))$

(For all situations, if a state of affairs is the case, then it is permitted by God.)

Proof.

1. $\mathcal{C}_\theta\neg\alpha(p) \to \neg\alpha(p)$	[T4, $\alpha(p)/\neg\alpha(p)$, Spec.]
2. $\mathcal{D}_\theta\alpha(p) \leftrightarrow \neg\mathcal{C}_\theta\neg\alpha(p)$	[T9, Spec]
3. $\alpha(p) \to \mathcal{D}_\theta\alpha(p)$	[**PC**, 1, 2]
4. $\forall p(\alpha(p) \to \mathcal{D}_\theta\alpha(p))$	[Gen, 3]

$\square$

Theorems T11 and T12, on the other hand, characterize the relation between God's opposition regarding states of affairs:

T11. $\forall p(\mathcal{S}_\theta\alpha(p) \to \neg\alpha(p))$

(For all situations, if God is opposed to a state of affairs, then such a state of affairs is not the case.)

Proof.

1. $\mathcal{C}_\theta\neg\alpha(p) \to \neg\alpha(p)$	[T4, $\alpha(p)/\neg\alpha(p)$, Spec]
2. $\mathcal{S}_\theta\alpha(p) \leftrightarrow \mathcal{C}_\theta\neg\alpha(p)$	[A8, Spec]
3. $\mathcal{S}_\theta\alpha(p) \to \neg\alpha(p)$	[**PC**, 1, 2]
4. $\forall p(\mathcal{S}_\theta\alpha(p) \to \neg\alpha(p))$	[Gen, 3]

$\square$

T12. $\forall p(\mathcal{S}_\theta\alpha(p) \to \mathcal{D}_\theta\neg\alpha(p))$

(For all situations, if God is opposed to a state of affairs, then He permits the opposite.)

Proof.

1. $\mathcal{S}_\theta\alpha(p) \to \neg\alpha(p)$	[T11, Spec]
2. $\neg\alpha(p) \to \mathcal{D}_\theta\neg\alpha(p)$	[T10, $\alpha(p)/\neg\alpha(p)$, Spec]
3. $\mathcal{S}_\theta\alpha(p) \to \mathcal{D}_\theta\neg\alpha(p)$	[**PC**, 1, 2]
4. $\forall p(\mathcal{S}_\theta\alpha(p) \to \mathcal{D}_\theta\neg\alpha(p))$	[Gen, 3]

$\square$

Theorems from T13 to T23 state some inner relations between will, opposition, and permission of God regarding states of affairs.

T13. $\forall p(\mathcal{C}_\theta \mathcal{C}_\theta \alpha(p) \leftrightarrow \mathcal{C}_\theta \alpha(p))$

(God wills to will a state of affairs *iff* He wills such a state of affairs.)

Proof.

1. $\mathcal{C}_\theta \mathcal{C}_\theta \alpha(p) \rightarrow \mathcal{C}_\theta \alpha(p)$ [T4, $\alpha(p)/\mathcal{C}_\theta \alpha(p)$, Spec]
2. $\mathcal{C}_\theta \alpha(p) \rightarrow \mathcal{C}_\theta \mathcal{C}_\theta \alpha(p)$ [A2, Spec]
3. $\mathcal{C}_\theta \mathcal{C}_\theta \alpha(p) \leftrightarrow \mathcal{C}_\theta \alpha(p)$ [**PC**, 1, 2]
4. $\forall p(\mathcal{C}_\theta \mathcal{C}_\theta \alpha(p) \leftrightarrow \mathcal{C}_\theta \alpha(p))$ [Gen, 3]

□

T14. $\forall p(\mathcal{C}_\theta \mathcal{D}_\theta \alpha(p) \leftrightarrow \mathcal{D}_\theta \alpha(p))$

(For all situations, God wills to permit a state of affairs *iff* He permits such a state of affairs.)

Proof.

1. $\mathcal{C}_\theta \mathcal{D}_\theta \alpha(p) \rightarrow \mathcal{D}_\theta \alpha(p)$ [T4, $\alpha(p)/\mathcal{D}_\theta \alpha(p)$, Spec]
2. $\mathcal{D}_\theta \alpha(p) \rightarrow \mathcal{C}_\theta \mathcal{D}_\theta \alpha(p)$ [A3, Spec]
3. $\mathcal{C}_\theta \mathcal{D}_\theta \alpha(p) \leftrightarrow \mathcal{D}_\theta \alpha(p)$ [**PC**, 1, 2]
3. $\forall p(\mathcal{C}_\theta \mathcal{D}_\theta \alpha(p) \leftrightarrow \mathcal{D}_\theta \alpha(p))$ [Gen, 3]

□

T15. $\forall p(\mathcal{D}_\theta \mathcal{D}_\theta \alpha(p) \leftrightarrow \mathcal{D}_\theta \alpha(p))$

(For all situations, God permits to permit a state of affairs *iff* He permits such a state of affairs.)

Proof.

1. $\mathcal{C}_\theta \mathcal{C}_\theta \neg\alpha(p) \leftrightarrow \mathcal{C}_\theta \neg\alpha(p)$ [T13, $\alpha(p)/\neg\alpha(p)$, Spec]
2. $\mathcal{D}_\theta \alpha(p) \leftrightarrow \neg\mathcal{C}_\theta \neg\alpha(p)$ [T9, Spec]
3. $\neg\mathcal{D}_\theta \alpha(p) \leftrightarrow \mathcal{C}_\theta \neg\alpha(p)$ [**PC**, 2]
4. $\mathcal{C}_\theta \neg\mathcal{D}_\theta \alpha(p) \leftrightarrow \neg\mathcal{D}_\theta \alpha(p)$ [Eq, 3 in 1]
5. $\neg\mathcal{D}_\theta \mathcal{D}_\theta \alpha(p) \leftrightarrow \neg\mathcal{D}_\theta \alpha(p)$ [Eq, 3 in 4]
6. $\mathcal{D}_\theta \mathcal{D}_\theta \alpha(p) \leftrightarrow \mathcal{D}_\theta \alpha(p)$ [**PC**, 5]
7. $\forall p(\mathcal{D}_\theta \mathcal{D}_\theta \alpha(p) \leftrightarrow \mathcal{D}_\theta \alpha(p))$ [Gen, 6]

□

The formula $\forall p(\mathcal{C}_\theta(\mathcal{C}_\theta\alpha(p)) \to \mathcal{D}_\theta(\mathcal{D}_\theta\alpha(p)))$ corresponds to a formula which is originally an axiom in the system of Nieznański. It is easily demonstrated from T6, A9, T13, and T15: [12]

T16. $\forall p(\mathcal{C}_\theta\mathcal{C}_\theta\alpha(p) \to \mathcal{D}_\theta\mathcal{D}_\theta\alpha(p))$

(For all situations, if God wills to will a state of affairs, then God permits to permit such a state of affairs.)

Proof.

1. $\mathcal{C}_\theta\alpha(p) \to \neg\mathcal{C}_\theta\neg\alpha(p)$ [T6, Spec]
2. $\mathcal{D}_\theta\alpha(p) \leftrightarrow \neg\mathcal{C}_\theta\neg\alpha(p)$ [T9, Spec]
3. $\mathcal{C}_\theta\alpha(p) \to \mathcal{D}_\theta\alpha(p)$ [Eq, 2 in 1]
4. $\mathcal{C}_\theta\mathcal{C}_\theta\alpha(p) \leftrightarrow \mathcal{C}_\theta\alpha(p)$ [T13, Spec]
5. $\mathcal{C}_\theta\mathcal{C}_\theta\alpha(p) \to \mathcal{D}_\theta\alpha(p)$ [Eq, 4 in 3]
6. $\mathcal{D}_\theta\mathcal{D}_\theta\alpha(p) \leftrightarrow \mathcal{D}_\theta\alpha(p)$ [T15, Spec]
7. $\mathcal{C}_\theta\mathcal{C}_\theta\alpha(p) \to \mathcal{D}_\theta\mathcal{D}_\theta\alpha(p)$ [Eq, 6 in 5]
8. $\forall p(\mathcal{C}_\theta\mathcal{C}_\theta\alpha(p) \to \mathcal{D}_\theta\mathcal{D}_\theta\alpha(p))$ [Gen, 7]

$\square$

T17. $\forall p(\mathcal{D}_\theta\mathcal{C}_\theta\alpha(p) \leftrightarrow \mathcal{C}_\theta\alpha(p))$

(For all situations, God permits to will a state of affairs *iff* He wills such a state of affairs.)

Proof.

1. $\mathcal{C}_\theta\mathcal{D}_\theta\neg\alpha(p) \leftrightarrow \mathcal{D}_\theta\neg\alpha(p)$ [T14, $\alpha(p)/\neg\alpha(p)$, Spec]
2. $\mathcal{D}_\theta\neg\alpha(p) \leftrightarrow \neg\mathcal{C}_\theta\alpha(p))$ [T9.2, Spec]
3. $\mathcal{C}_\theta\neg\mathcal{C}_\theta\alpha \leftrightarrow \neg\mathcal{C}_\theta\alpha$ [Eq, 2 in 1]
4. $\neg\mathcal{D}_\theta\mathcal{C}_\theta\alpha(p) \leftrightarrow \mathcal{C}_\theta\neg\mathcal{C}_\theta\alpha(p))$ [T9.1, $\alpha(p)/\mathcal{C}_\theta\alpha(p)$, Spec]
5. $\neg\mathcal{D}_\theta\mathcal{C}_\theta\alpha(p) \leftrightarrow \neg\mathcal{C}_\theta\alpha(p))$ [Eq, 3 in 4]
6. $\mathcal{D}_\theta\mathcal{C}_\theta\alpha(p) \leftrightarrow \mathcal{C}_\theta\alpha(p)$ [**PC**, 5]
7. $\forall p(\mathcal{D}_\theta\mathcal{C}_\theta\alpha(p) \leftrightarrow \mathcal{C}_\theta\alpha(p))$ [Gen, 6]

$\square$

T18. $\forall p(\mathcal{C}_\theta\mathcal{S}_\theta\alpha(p) \leftrightarrow \mathcal{S}_\theta\alpha(p))$

(For all situations, God wills to oppose a state of affairs *iff* He is opposed to such a state of affairs.)

[12] Nieznański called the axiom associated with this theorem "Axiom of justice" [11, p. 208]. The original version was quite different of that in this paper, for it quantifies over modal operators and we have avoided to do this throughout **N1**. Written in our notation, it would be $\forall p(\exists x\mathcal{C}_\theta\mathcal{C}_x\alpha(p) \to \forall x(\mathcal{D}_\theta\mathcal{D}_\theta\alpha(p)))$.

Proof.
1. $\mathcal{C}_\theta\mathcal{C}_\theta\neg\alpha(p) \leftrightarrow \mathcal{C}_\theta\neg\alpha(p)$ [T13, $\alpha(p)/\neg\alpha(p)$, Spec]
2. $\mathcal{S}_\theta\alpha(p) \leftrightarrow \mathcal{C}_\theta\neg\alpha(p)$ [A8, Spec]
3. $\mathcal{C}_\theta\mathcal{S}_\theta\alpha(p) \leftrightarrow \mathcal{S}_\theta\alpha(p)$ [Eq, 2 in 1]
4. $\forall p(\mathcal{C}_\theta\mathcal{S}_\theta\alpha(p) \leftrightarrow \mathcal{S}_\theta\alpha(p))$ [Gen, 3]

□

T19. $\forall p(\mathcal{D}_\theta\mathcal{S}_\theta\alpha(p) \leftrightarrow \mathcal{S}_\theta\alpha(p))$

(For all situations, God permits to oppose a state of affairs *iff* He opposes such a state of affairs.)

Proof.
1. $\mathcal{D}_\theta\mathcal{C}_\theta\neg\alpha(p) \leftrightarrow \mathcal{C}_\theta\neg\alpha(p)$ [T17, $\alpha(p)/\neg\alpha(p)$, Spec]
2. $\mathcal{S}_\theta\alpha(p) \leftrightarrow \neg\mathcal{C}_\theta\alpha(p)$ [A8, Spec]
3. $\mathcal{D}_\theta\mathcal{S}_\theta\alpha(p) \leftrightarrow \mathcal{S}_\theta\alpha(p)$ [Eq, 2 in 1]
4. $\forall p(\mathcal{D}_\theta\mathcal{S}_\theta\alpha(p) \leftrightarrow \mathcal{S}_\theta\alpha(p))$ [Gen, 3]

□

T20. $\forall p(\mathcal{S}_\theta\mathcal{D}_\theta\alpha(p) \leftrightarrow \mathcal{S}_\theta\alpha(p))$

(For all situations, God opposes to permit a state of affairs *iff* He opposes such a state of affairs.)

Proof.
1. $\mathcal{D}_\theta\mathcal{D}_\theta\alpha(p) \leftrightarrow \mathcal{D}_\theta\alpha(p))$ [T15, Spec]
2. $\neg\mathcal{D}_\theta\mathcal{D}_\theta\alpha(p) \leftrightarrow \neg\mathcal{D}_\theta\alpha(p)$ [**PC**, 1]
3. $\mathcal{S}_\theta\alpha(p) \leftrightarrow \neg\mathcal{D}_\theta\alpha(p)$ [T8, Spec]
4. $\mathcal{S}_\theta\mathcal{D}_\theta\alpha(p) \leftrightarrow \mathcal{S}_\theta\alpha(p)$ [Eq, 3 in 2]
5. $\forall p(\mathcal{S}_\theta\mathcal{D}_\theta\alpha(p) \leftrightarrow \mathcal{S}_\theta\alpha(p))$ [Gen, 4]

□

T21. $\forall p(\mathcal{S}_\theta\mathcal{C}_\theta\alpha(p) \leftrightarrow \mathcal{D}_\theta\neg\alpha(p))$

(For all situations, God opposes to will a state of affairs *iff* He permits the state of affairs not to be the case.)

Proof.
1. $\mathcal{C}_\theta\mathcal{D}_\theta\neg\alpha(p) \leftrightarrow \mathcal{D}_\theta\neg\alpha(p)$ [T14, $\alpha(p)/\neg\alpha(p)$, Spec]
2. $\mathcal{D}_\theta\neg\alpha(p) \leftrightarrow \neg\mathcal{C}_\theta\alpha(p)$ [T9.2, Spec.]
3. $\mathcal{C}_\theta\neg\mathcal{C}_\theta\alpha(p) \leftrightarrow \mathcal{D}_\theta\neg\alpha(p)$ [Eq, 2 in 1]
4. $\mathcal{S}_\theta\mathcal{C}_\theta\alpha(p) \leftrightarrow \mathcal{C}_\theta\neg\mathcal{C}_\theta\alpha(p)$ [A8, $\alpha(p)/\mathcal{C}_\theta\alpha(p)$, Spec]
5. $\mathcal{S}_\theta\mathcal{C}_\theta\alpha(p) \leftrightarrow \mathcal{D}_\theta\neg\alpha(p)$ [Eq, 4 in 3]
6. $\forall p(\mathcal{S}_\theta\mathcal{C}_\theta\alpha(p) \leftrightarrow \mathcal{D}_\theta\neg\alpha(p))$ [Gen, 5]

□

T22. $\forall p(\mathcal{S}_\theta\mathcal{S}_\theta\alpha(p) \leftrightarrow \mathcal{D}_\theta\alpha(p))$

(For all situations, God opposes to oppose a state of affairs *iff* He permits such a state of affairs.)

Proof.

1. $\mathcal{S}_\theta\mathcal{S}_\theta\alpha(p) \leftrightarrow \mathcal{C}_\theta\neg\mathcal{S}_\theta\alpha(p)$	[A8, $\alpha(p)/\mathcal{S}_\theta\alpha(p)$, Spec]
2. $\mathcal{S}_\theta\alpha(p) \leftrightarrow \neg\mathcal{D}_\theta\alpha(p)$	[T8, Spec]
3. $\mathcal{S}_\theta\mathcal{S}_\theta\alpha(p) \leftrightarrow \mathcal{C}_\theta\neg\neg\mathcal{D}_\theta\alpha(p)$	[Eq, 2 in 1]
4. $\mathcal{S}_\theta\mathcal{S}_\theta\alpha(p) \leftrightarrow \mathcal{C}_\theta\mathcal{D}_\theta\alpha(p)$	[**PC**, 3]
5. $\mathcal{C}_\theta\mathcal{D}_\theta\alpha(p) \leftrightarrow \mathcal{D}_\theta\alpha(p)$	[T14, Spec]
6. $\mathcal{S}_\theta\mathcal{S}_\theta\alpha(p) \leftrightarrow \mathcal{D}_\theta\alpha(p)$	[Eq, 5 in 4]
7. $\forall p(\mathcal{S}_\theta\mathcal{S}_\theta\alpha(p) \leftrightarrow \mathcal{D}_\theta\alpha(p))$	[Gen, 6]

□

T23. $\forall p(\mathcal{C}_\theta\alpha(p) \rightarrow \neg\mathcal{C}_\theta\mathcal{C}_\theta\neg\alpha(p))$

(For all situations, if God wills a state of affairs, then He does not will to will the opposite.)

Proof.

1. $\mathcal{C}_\theta\alpha(p) \rightarrow \alpha(p)$	[T4, Spec]
2. $\alpha(p) \rightarrow \mathcal{D}_\theta\alpha(p)$	[T10, Spec]
3. $\mathcal{C}_\theta\alpha(p) \rightarrow \mathcal{D}_\theta\alpha(p)$	[**PC**, 1, 2]
4. $\mathcal{D}_\theta\mathcal{D}_\theta\alpha(p) \leftrightarrow \mathcal{D}_\theta\alpha(p)$	[T15, Spec]
5. $\neg\mathcal{C}_\theta\neg\mathcal{D}_\theta\alpha(p) \leftrightarrow \mathcal{D}_\theta\mathcal{D}_\theta\alpha(p)$	[T9, $\alpha(p)/\mathcal{D}_\theta\alpha(p)$, Spec]
6. $\neg\mathcal{C}_\theta\neg\mathcal{D}_\theta\alpha(p) \leftrightarrow \mathcal{D}_\theta\alpha(p)$	[Eq, 5 in 4]
7. $\neg\mathcal{D}_\theta\alpha(p) \leftrightarrow \mathcal{C}_\theta\neg\alpha(p)$	[T9.1, Spec]
8. $\neg\mathcal{C}_\theta\mathcal{C}_\theta\neg\alpha(p) \leftrightarrow \mathcal{D}_\theta\alpha(p)$	[Eq, 7 in 6]
9. $\mathcal{C}_\theta\alpha(p) \rightarrow \neg\mathcal{C}_\theta\mathcal{C}_\theta\neg\alpha(p)$	[Eq, 8 in 3]
10. $\forall p(\mathcal{C}_\theta\alpha(p) \rightarrow \neg\mathcal{C}_\theta\mathcal{C}_\theta\neg\alpha(p))$	[Gen, 9]

□

3 God and values: a theistic axiology

In this section, we deal with a formal axiology, i. e., a formal treatment of our ordinary notions of "good", "evil", and "neutral" situations. Good, evil, and neutral situations are established in the axioms A5, A6, A10, and A11.

Theorems from T24 to T31 are consequences of such axioms. They show some of the relations between good, evil, and neutral situations and their opposites.

T24. $\forall p(\neg n(p) \leftrightarrow (d(p) \vee z(p)))$

(A situation is not neutral *iff* either it is good or evil.)

Proof.
1. $n(p) \leftrightarrow (\neg d(p) \wedge \neg z(p))$ [A11, Spec]
2. $\neg n(p) \leftrightarrow \neg(\neg d(p) \wedge \neg z(p))$ [**PC**, 1]
3. $\neg n(p) \leftrightarrow (\neg\neg d(p) \vee \neg\neg z(p))$ [**PC**, 2]
4. $\neg n(p) \leftrightarrow (d(p) \vee z(p))$ [**PC**, 3]
5. $\forall p(\neg n(p) \leftrightarrow (d(p) \vee z(p)))$ [Gen, 4]

□

T25. $\forall p(n(p) \vee d(p) \vee z(p))$

(Every situation is neutral, good, or evil.)

Proof.
1. $\neg n(p) \leftrightarrow (d(p) \vee z(p))$ [T24, Spec]
2. $\neg n(p) \rightarrow (d(p) \vee z(p))$ [**PC**, 1]
3. $n(p) \vee (d(p) \vee z(p))$ [Def. 3, 2]
4. $n(p) \vee d(p) \vee z(p)$ [**PC**, 3]
5. $\forall p(n(p) \vee d(p) \vee z(p))$ [Gen, 4]

□

T26. $\forall p \forall q(Op(p,q) \rightarrow (d(p) \rightarrow \neg d(q)))$

(For all situations, if two situations are opposite, then if one is good, the other is not good.)

Proof.
1. $Op(p,q)$ [Hip.]
2. $Op(p,q) \rightarrow (d(p) \leftrightarrow z(q))$ [A7, Spec]
3. $d(p) \leftrightarrow z(q)$ [MP, 1, 2]
4. $z(q) \rightarrow \neg d(q)$ [A6, Spec]
5. $d(p) \rightarrow \neg d(q)$ [Eq, 3 in 4]
6. $Op(p,q) \rightarrow (d(p) \rightarrow \neg d(q))$ [DT, 1–5]
7. $\forall p \forall q(Op(p,q) \rightarrow (d(p) \rightarrow \neg d(q)))$ [Gen, 7]

□

T27. $\forall p(d(p) \rightarrow \neg z(p))$

(For all situations, if a situation is good, then it is not evil.)

Proof.

1. $z(p) \to \neg d(p)$ [A6, Spec]
2. $d(p) \to \neg z(p)$ [**PC**, 1]
3. $\forall p(d(p) \to \neg z(p))$ [Gen, 2]

□

T28. $\forall p(\neg d(p) \to (n(p) \vee z(p)))$

(For all situations, if a situation is not good, then it is neutral or evil.)

Proof.

1. $n(p) \vee d(p) \vee z(p)$ [T25, Spec]
2. $d(p) \vee (n(p) \vee z(p))$ [**PC**, 1]
3. $\neg d(p) \to (n(p) \vee z(p))$ [**PC**, 2]
4. $\forall p(\neg d(p) \to (n(p) \vee z(p)))$ [Gen, 4]

□

T29. $\forall p(((n(p) \vee z(p)) \to \neg d(p)))$

(For all situations, if a situation is neutral or evil, then it is not good.)

Proof.

1. $\neg((n(p) \vee z(p)) \to \neg d(p))$ [Hip.]
2. $\quad (n(p) \vee z(p)) \wedge \neg\neg d(p)$ [**PC**, 1]
3. $\quad n(p) \vee z(p)$ [**PC**, 2]
4. $\quad d(p)$ [**PC**, 2]
5. $\qquad z(p)$ [Hip., 3]
6. $\qquad z(p) \to \neg d(p)$ [A6, Spec]
7. $\qquad \neg d(p)$ [MP, 5, 6]
8. $\qquad n(p)$ [Hip., 3]
9. $\qquad n(p) \leftrightarrow (\neg d(p) \wedge \neg z(p))$ [A11, Spec]
10. $\qquad (\neg d(p) \wedge \neg z(p))$ [MP, 8, 9]
11. $\qquad \neg d(p)$ [**PC**, 10]
12. $\quad \neg d(p)$ [5-11]
13. $\neg\neg((n(p) \vee z(p)) \to \neg d(p))$ [¬ Hip, 1, 4, 12]
14. $((n(p) \vee z(p)) \to \neg d(p))$ [**PC**, 13]
15. $\forall p((n(p) \vee z(p)) \to \neg d(p))$ [Gen, 14]

□

T30. $\forall p(\neg d(p) \leftrightarrow (n(p) \vee z(p)))$

(A situation is not good *iff* either it is neutral or evil.)

Proof.
1. $\neg d(p) \to (n(p) \vee z(p))$ [T28]
2. $((n(p) \vee z(p)) \to \neg d(p))$ [T29]
3. $\neg d(p) \leftrightarrow (n(p) \vee z(p))$ [**PC**, 1, 2]
4. $\forall p(\neg d(p) \leftrightarrow (n(p) \vee z(p)))$ [Gen, 3]

$\square$

T31. $\forall p \forall q(Op(p, q) \to (n(p) \leftrightarrow n(q)))$

(If two situations are opposite, then one of them is neutral *iff* the other is also neutral.)

Proof.
1. $Op(p, q)$ [Hip]
2. $Op(p, q) \to (d(p) \leftrightarrow z(q))$ [A7, Spec]
3. $d(p) \leftrightarrow z(q)$ [MP, 1, 2]
4. $n(p) \leftrightarrow (\neg d(p) \wedge \neg z(p))$ [A11, Spec]
5. $n(q) \leftrightarrow (\neg d(q) \wedge \neg z(q))$ [A11, p/q, Spec]
6. $d(q) \leftrightarrow z(p)$ [US, 3]
7. $n(p) \leftrightarrow (\neg z(q) \wedge \neg d(q))$ [Eq, 3 & 6 in 4]
8. $n(p) \leftrightarrow (\neg d(q) \wedge \neg z(q))$ [**PC**, 7]
9. $n(p) \leftrightarrow n(q)$ [**PC**, 5, 8]
10. $Op(p, q) \to (n(p) \leftrightarrow n(q))$ [DT, 1–9]
11. $\forall p \forall q(Op(p, q) \to (n(p) \leftrightarrow n(q)))$ [Gen, 10]

$\square$

T32. $\forall p(\mathcal{C}_\theta P(p) \to d(p))$

(For all situations, if God wills a situation to be the case, then such a situation is good.)

Proof.
1. $d(p) \leftrightarrow \mathcal{C}_\theta P(p)$ [A5, Spec]
2. $\mathcal{C}_\theta P(p) \to d(p)$ [**PC**, 1]
3. $\forall p(\mathcal{C}_\theta P(p) \to d(p))$ [Gen, 2]

$\square$

T33. $\forall p(\mathcal{S}_\theta P(p) \to \neg d(p))$

(For all situations, if God opposes to a situation that is the case, then such a situation is not good.)

Proof.
1. $d(p) \leftrightarrow \mathcal{C}_\theta P(p)$ [A5]
2. $\neg d(p) \leftrightarrow \neg\mathcal{C}_\theta P(p)$ [**PC**, 1]
3. $\mathcal{C}_\theta P(p) \to \neg\mathcal{C}_\theta\neg P(p)$ [T6, $\alpha(p)/P(p)$), Spec]
4. $\mathcal{C}_\theta\neg P(p) \to \neg\mathcal{C}_\theta P(p)$ [**PC**, 3]
5. $\mathcal{C}_\theta\neg P(p) \to \neg d(p)$ [Eq, 2 in 4]
6. $\mathcal{S}_\theta P(p) \leftrightarrow \mathcal{C}_\theta\neg P(p)$ [A8, $\alpha(p)/P(p)$, Spec]
7. $\mathcal{S}_\theta P(p) \to \neg d(p)$ [Eq, 6 in 5]
8. $\forall p(\mathcal{S}_\theta P(p) \to \neg d(p))$ [Gen, 7]

□

It is easy to prove the following theorem from A10, A11, and T33:

T34. $\forall p(\mathcal{S}_\theta P(p) \to z(p))$

(For all situations, if God opposes to a situation to be the case, then the situation is evil.)

The following theorems T35, T36, and T37 establish the relation between the axiological values of situations and the permission of God.

T35. $\forall p(d(p) \to \mathcal{D}_\theta P(p))$

(For all situations, if a situation is good, then God permits it to be the case.)

Proof.
1. $\mathcal{S}_\theta P(p) \to \neg d(p)$ [T33, Spec.]
2. $d(p) \to \neg\mathcal{S}_\theta P(p)$ [**PC**, 1]
3. $\mathcal{D}_\theta P(p) \leftrightarrow \neg\mathcal{S}_\theta P(p)$ [A9, $\alpha(p)/\alpha(p)$, Spec]
4. $d(p) \to \mathcal{D}_\theta P(p)$ [Eq, 3 in 2]
5. $\forall p(\to \mathcal{D}_\theta P(p))$ [Gen, 4]

□

T36. $\forall p(\neg d(p) \to \mathcal{D}_\theta\neg P(p))$

(For all situations, if a situation is not good, then God permits it not to be the case.)

Proof.
1. $\mathcal{C}_\theta\alpha(p) \to d(p))$ [T32, Spec]
2. $\neg d(p) \to \neg\mathcal{C}_\theta P(p)$ [**PC**, 1]
4. $\neg\mathcal{C}_\theta P(p) \leftrightarrow \mathcal{D}_\theta\neg P(p)$ [T9.2, $\alpha(p)/P(p)$, Spec]
5. $\neg d(p) \to \mathcal{D}_\theta\neg P(p)$ [Eq, 4 in 3]
6. $\forall p(\neg d(p) \to \mathcal{D}_\theta\neg P)$ [Gen, 5]

□

T37. $\forall p(n(p) \to (\mathcal{D}_\theta P(p) \wedge \mathcal{D}_\theta \neg P(p)))$

(For all situations, if a situation is neutral, then God permits it to be or not to be the case.)

Proof.
1. $n(p) \leftrightarrow (\neg d(p) \wedge \neg z(p))$ [A11, Spec]
2. $\neg z(p) \to \neg \mathcal{S}_\theta P(p)$ [T35, Spec, **PC**]
3. $\mathcal{S}_\theta P(p) \leftrightarrow \neg \mathcal{D}_\theta P(p)$ [A9, $\alpha(p)/P(p)$, Spec]
4. $\neg z(p) \to \mathcal{D}_\theta P(p)$ [Eq, 2 in 3, **PC**]
5. $\neg d(p) \to \mathcal{D}_\theta \neg P(p)$ [T36, Spec]
6. $n(p) \to (\mathcal{D}_\theta P(p) \wedge \mathcal{D}_\theta \neg P(p))$ [**PC**, 1, 4, 5]
7. $\forall p(n(p) \to (\mathcal{D}_\theta P(p) \wedge \mathcal{D}_\theta \neg P(p)))$ [Gen, 4]

□

The latter theorem states that some situations, namely neutral situations, are such that both their occurrence and non-occurrence are permitted by God.

The results established so far allow us to address the problem of determinism.

4 Refutation of determinism

As stated earlier, our main concern is to give an answer to the following determinist claim:

$$\textbf{(DET1)}\ \forall p(P(p) \to \mathcal{C}_\theta P(p))$$

We are now in position to answer this claim through formal means. The axiom A4 can be informally interpreted as saying that “not everything is flowers” in the world, or, as stated by Nieznański, “not all events are good” [11, p. 211.]:

A4. $\neg\forall p(P(p) \to d(p))$

In **N1**, however, we derive T38, a very important theorem, since that it is the negation of **DET1**:

T38 ($\neg$**DET1**)**.** $\neg\forall p(P(p) \to \mathcal{C}_\theta P(p))$

(Not all situations is such that, if a situation is the case, then God wills such a situation to be the case.)

Proof.
1. $\neg\neg\forall p(P(p) \to \mathcal{C}_\theta P(p))$ [Hip]

2. $\forall p(P(p) \to \mathcal{C}_\theta P(p))$ [**PC**, 1]
3. $P(p) \to \mathcal{C}_\theta P(p)$ [2, Spec]
4. $\forall p(\mathcal{C}_\theta P(p) \to d(p))$ [T32]
5. $\mathcal{C}_\theta P(p) \to d(p)$ [Spec, 4]
6. $P(p) \to d(p)$ [**PC**, 3, 5]
7. $\forall p(P(p) \to d(p))$ [Gen, 6]
8. $\neg\forall p(P(p) \to d(p)$ [A4]
9. $\neg\neg\neg\forall p(P(p) \to \mathcal{C}_\theta P(p))$ [$\neg$ Hip, 7, 8]
10. $\neg\forall p(P(p) \to \mathcal{C}_\theta P(p))$ [**PC**, 9]

$\square$

In what follows, it is defined what it means for God to be a 'want-it-all', the kind of person that always wills some state of affairs.

Def. 10 (Want-it-all)**.** $OW :\leftrightarrow \forall p(\mathcal{C}_\theta\alpha(p) \vee \mathcal{C}_\theta\neg\alpha(p))$

(God is a 'want-it-all' regarding situations *iff* for all situations God wills a state of affairs or its opposite.)

The following theorem shows that, in **N1**, God is not a 'want-it-all'.

T39. $\neg OW$

(God is not a 'want-it-all')

Proof.
1. $\forall p(\mathcal{C}_\theta\alpha(p) \vee \mathcal{C}_\theta\neg\alpha(p))$ [Hip]
2. $\forall p(\neg\mathcal{C}_\theta\alpha(p) \to \mathcal{C}_\theta\neg\alpha(p))$ [**PC**, 1]
3. $\neg\mathcal{C}_\theta\alpha(p) \to \mathcal{C}_\theta\neg\alpha(p)$ [Spec, 2]
4. $\mathcal{C}_\theta\neg\alpha(p) \to \neg\alpha(p)$ [T4, $\alpha(p)/\neg\alpha(p)$]
5. $\neg\mathcal{C}_\theta\alpha(p) \to \neg\alpha(p)$ [**PC**, 3, 4]
6. $\forall p(\neg\mathcal{C}_\theta\alpha(p) \to \neg\alpha(p))$ [Gen, 5]
7. $\forall p(\alpha(p) \to \mathcal{C}_\theta\alpha(p))$ [**PC**, 6]
8. $\neg\forall p(\alpha(p) \to \mathcal{C}_\theta\alpha(p))$ [T38]
9. $\neg\forall p(\mathcal{C}_\theta\alpha(p) \vee \mathcal{C}_\theta\neg\alpha(p))$ [$\neg$ Hip, 1]
10. $\neg OW$ [Def. 10, 9]

$\square$

Another statement of interest here is the following:

$$\textbf{(DET2)}\ \forall p(\mathcal{W}_\theta\alpha(p) \to \mathcal{C}_\theta\alpha(p))$$

It is a remarkable fact that, in **N1**, **DET1** and **DET2** are equivalent, as T40 shows:

T40 (**DET2** $\leftrightarrow$ **DET1**). $\forall p(\mathcal{W}_\theta P(p) \rightarrow \mathcal{C}_\theta P(p)) \leftrightarrow \forall p(P(p) \rightarrow \mathcal{C}_\theta P(p))$

(For all situations, to affirm that if God knows a situation to be the case, then God wills such a situation to be the case, is equivalent to affirm that if a situation is the case, then God wills it to be the case.)

Proof.
1. $P(p) \rightarrow \mathcal{W}_\theta P(p)$ [T2, $\alpha(p)/P(p)$, Spec]
2. $\mathcal{W}_\theta P(p) \rightarrow P(p))$ [T3, $\alpha(p)/P(p)$, Spec]
3. $\mathcal{W}_\theta P(p) \leftrightarrow P(p)$ [**PC**, 1, 2]
4. $(P(p) \rightarrow \mathcal{C}_\theta P(p)) \leftrightarrow (P(p) \rightarrow \mathcal{C}_\theta P(p))$ [**PC**-Theorem]
5. $(\mathcal{W}_\theta P(p) \rightarrow \mathcal{C}_\theta P(p)) \leftrightarrow (P(p) \rightarrow \mathcal{C}_\theta P(p))$ [Eq, 3 in 4]
6. $\forall p(\mathcal{W}_\theta P(p) \rightarrow \mathcal{C}_\theta P(p)) \leftrightarrow \forall p(P(p) \rightarrow \mathcal{C}_\theta P(p))$ [Gen, 5]

□

But **DET1** is false, thus, from T40, **DET2** is also false:

T41 ($\neg$ **DET2**). $\neg\forall p(\mathcal{W}_\theta P(p) \rightarrow \mathcal{C}_\theta P(p))$ □

(Not all situations are such that if God knows a situation to be the case, then God wills such a situation to be the case.)

The definition that follows sets up a new operator, and the theorems that follow extend the meaning of some results just stated above. We interpret it as 'God is the cause of':[13]

Def. 11 (God is the direct cause of). $(\mathcal{A}_\theta \alpha(p) :\leftrightarrow \mathcal{C}_\theta \alpha(p))$

(God is the direct cause of a state of affairs *iff* He wills such a state of affairs.)

The following two theorems establish the relation between God as direct cause of situations and situations that are the case.

T42. $\forall p(\mathcal{A}_\theta \alpha(p) \rightarrow \alpha(p))$

(For all situations, if God is the direct cause of a state of affairs, then such a state of affairs is the case.)

[13] Although recognizing Nieznański's merit on defining this operator and its meaning in the context of a formal theodicy (as an attempt to deal with the will of God, His responsibility and the fact that He is the cause of everything in some sense), we interpret it in a different way: instead of interpreting the operator defined in what follows as 'God is the cause of', 'God is the *direct* cause of', for God's will is effective. Another relevant difference is that, in **N1**, the only person involved is God, and by doing this we avoid problems with quantifiers and multi-modalities – for instance, the definition above in his system would be stated as $\mathcal{A}_x \alpha(p) :\leftrightarrow \mathcal{C}_x \alpha(p)$, where x can be quantified.

Proof.
1. $\mathcal{A}_\theta\alpha(p) \leftrightarrow \mathcal{C}_\theta\alpha(p)$ [Def. 11]
2. $\mathcal{C}_\theta\alpha(p) \to \alpha(p)$ [T4, Spec]
3. $\mathcal{A}_\theta\alpha(p) \to \alpha(p)$ [Eq, 1 in 2]
4. $\forall p(\mathcal{A}_\theta\alpha(p) \to \alpha(p))$ [Gen, 3]

□

T43. $\neg\forall p(P(p) \to \mathcal{A}_\theta P(p))$

(God is not the direct cause of every situation that is the case.)

Proof.
1. $\forall p(P(p) \to \mathcal{A}_\theta P(p))$ [Hip]
2. $P(p) \to \mathcal{A}_\theta P(p)$ [Spec, 1]
3. $\mathcal{A}_\theta P(p) \leftrightarrow \mathcal{C}_\theta P(p)$ [Def. 11, $\alpha(p)/P(p)$]
4. $P(p) \to \mathcal{C}_\theta P(p)$ [Eq, 3 in 2]
5. $\forall p(P(p) \to \mathcal{C}_\theta P(p))$ [Gen, 4]
6. $\neg\forall p(P(p) \to \mathcal{C}_\theta P(p))$ [T38]
7. $\neg\forall p(P(p) \to \mathcal{A}_\theta P(p))$ [¬ Hip, 5, 6]

□

Next, we introduce the definition of contingent situation, that is a situation such that God permits it to be the case or not to be the case. Theorems from T44 to T49 show the relation between the will of God and contingent situations, and as a result, they show that there are contingent situations:

Def. 12 (Contingency)**.** $K(p) :\leftrightarrow (\mathcal{D}_\theta P(p) \wedge \mathcal{D}_\theta\neg P(p))$

(A situation is contingent *iff* God permits it to be or not to be the case.)

T44. $\forall p(K(p) \leftrightarrow (\neg\mathcal{C}_\theta P(p) \wedge \neg\mathcal{C}_\theta\neg P(p)))$

(For all situations, a situation is contingent *iff* neither God wills that situation to be the case, nor wills its opposite to be the case.)

Proof.
1. $K(p) \leftrightarrow (\mathcal{D}_\theta P(p) \wedge \mathcal{D}_\theta\neg P(p))$ [Def. 12]
2. $\mathcal{D}_\theta P(p) \leftrightarrow \neg\mathcal{C}_\theta\neg P(p)$ [T9, Spec]
3. $\mathcal{D}_\theta\neg P(p) \leftrightarrow \neg\mathcal{C}_\theta P(p)$ [T9.2, Spec]
4. $K(p) \leftrightarrow (\neg\mathcal{C}_\theta\neg P(p) \wedge \neg\mathcal{C}_\theta P(p))$ [Eq, 2 & 3 in 1]
5. $K(p) \leftrightarrow (\neg\mathcal{C}_\theta P(p) \wedge \neg\mathcal{C}_\theta\neg P(p))$ [**PC**, 4]
6. $\forall p(K(p) \leftrightarrow (\neg\mathcal{C}_\theta P(p) \wedge \neg\mathcal{C}_\theta\neg P(p)))$ [Gen, 5]

□

T45. $\forall p(K(p) \leftrightarrow (\neg \mathcal{C}_\theta P(p) \wedge \neg \mathcal{S}_\theta P(p)))$

(For all situations, a situation is contingent *iff* neither God wills that situation to be the case, nor is opposed to that.)

Proof.
1. $K(p) \leftrightarrow (\neg \mathcal{C}_\theta P(p) \wedge \neg \mathcal{C}_\theta \neg P(p))$ [T44, Spec]
2. $\mathcal{S}_\theta P(p) \leftrightarrow \mathcal{C}_\theta \neg P(p)$ [A8, Spec]
3. $\neg \mathcal{C}_\theta \neg P(p) \leftrightarrow \neg \mathcal{S}_\theta P(p)$ [**PC**, 2]
4. $K(p) \leftrightarrow (\neg \mathcal{C}_\theta P(p) \wedge \neg \mathcal{S}_\theta P(p))$ [Eq, 3 in 1]
5. $\forall p(K(p) \leftrightarrow (\neg \mathcal{C}_\theta P(p) \wedge \neg \mathcal{S}_\theta P(p)))$ [Gen, 4]

□

T46. $\forall p(K(p) \leftrightarrow \neg(\mathcal{C}_\theta P(p) \vee \mathcal{S}_\theta P(p)))$

(For all situations, a situation is contingent *iff* it is not the case that God wills that situation to be the case or He is opposed to that.)

Proof.
1. $K(p) \leftrightarrow (\neg \mathcal{C}_\theta P(p) \wedge \neg \mathcal{S}_\theta P(p))$ [T45, Spec]
2. $K(p) \leftrightarrow \neg(\mathcal{C}_\theta P(p) \vee \mathcal{S}_\theta P(p))$ [**PC**, 1]
3. $\forall p(K(p) \leftrightarrow \neg(\mathcal{C}_\theta P(p) \vee \mathcal{S}_\theta P(p)))$ [Gen, 2]

□

T47. $\exists p K(p) \leftrightarrow \neg \forall p(\mathcal{C}_\theta P(p) \vee \mathcal{S}_\theta P(p))$

(There is a contingent situation *iff* it is not the case that, for all situations, God wills that situation to be the case or He is opposed to that.)

Proof.
1. $K(p) \leftrightarrow \neg(\mathcal{C}_\theta P(p) \vee \mathcal{S}_\theta P(p))$ [T46, Spec]
2. $\exists p K(p) \leftrightarrow \exists p \neg(\mathcal{C}_\theta P(p) \vee \mathcal{S}_\theta P(p))$ [$\exists$, 2]
3. $\exists p K(p) \leftrightarrow \neg \forall p \neg\neg(\mathcal{C}_\theta P(p) \vee \mathcal{S}_\theta P(p))$ [Def. 1]
4. $\exists p K(p) \leftrightarrow \neg \forall p(\mathcal{C}_\theta P(p) \vee \mathcal{S}_\theta P(p))$ [**PC**, 3]

□

T48. $\exists p K(p) \leftrightarrow \neg OW$

(There is a contingent situation *iff* God is not a 'want-it-all'.)

Proof.
1. $\exists p K(p) \leftrightarrow \neg \forall p(\mathcal{C}_\theta P(p) \vee \mathcal{S}_\theta P(p))$ [T47]
2. $\mathcal{S}_\theta P(p) \leftrightarrow \mathcal{C}_\theta \neg P(p)$ [A8, $\alpha(p)/P(p)$]
3. $\exists p K(p) \leftrightarrow \neg \forall p(\mathcal{C}_\theta P(p) \vee \mathcal{C}_\theta \neg P(p))$ [Eq, 2 in 1]
4. $OW \leftrightarrow \forall p(\mathcal{C}_\theta P(p) \vee \mathcal{C}_\theta \neg P(p))$ [Def. 10]

5. $\exists pK(p) \leftrightarrow \neg OW$ [Eq, 4 in 3]

$\square$

T49. $\exists pK(p)$

(There is at least one situation that is contingent.)

Proof.
1. $\exists pK(p) \leftrightarrow \neg OW$ [T48]
2. $\neg OW$ [T39]
3. $\exists pK(p)$ [MP, 1, 2]

$\square$

An attempt to formalize the intuitive notion of responsibility is made below, where 'to be responsible for' is defined as an operator, $\mathcal{O}_\theta$. [14]

Def. 13 (Responsibility)**.** $\mathcal{O}_\theta\alpha(p) :\leftrightarrow \mathcal{A}_\theta\alpha(p)$

(God is responsible for a state of affairs *iff* He is the direct cause of that.)

Theorem T50 is simply the generalization of definition above:

T50. $\forall p(\mathcal{O}_\theta\alpha(p) \leftrightarrow \mathcal{A}_\theta\alpha(p))$

(For all situations, God is responsible for a state of affairs *iff* He is the direct cause of such a state of affairs.)

In the following last three theorems of **N1**, it is shown that if God is responsible for some situation, then it is good. But if some situation is evil, God is not responsible for. And, finally, if some evil happens, but God does not oppose to it (what would imply that it would not be the case), then the situation is contingent.

T51. $\forall p(\mathcal{O}_\theta P(p) \rightarrow d(p))$

(For all situations, if God is responsible for a situation that is the case, then the situation is good.)

[14] Originally, definition 13 was stated by Nieznański as $\mathcal{O}_x\alpha(p) :\leftrightarrow (\mathcal{A}_x\alpha(p) \vee (\neg\mathcal{S}_x\alpha(p) \wedge \mathcal{W}_x\mathcal{C}_\theta\mathcal{S}_x\alpha(p)))$, in the notation of this work. We recognize the merits of Nieznański's intuition: according to his thought, some person can be said "responsible" for a state of affairs whenever this person is the cause of that, or the person is not opposed to it, although knowing that God wills that person to be opposed to this state of affairs [11, p. 213]. But here, changing to θ all occurrences of x avoided problems with multi-modalities. This led as consequence to a simplification of the definition of responsibility, for God is the only person "formalized" in the system.

Proof.

1. $\mathcal{C}_\theta P(p) \to d(p))$ [T32, Spec]
2. $\mathcal{A}_\theta P(p) \leftrightarrow \mathcal{C}_\theta P$ [Def. 11, $\alpha(p)/P(p)$]
3. $\mathcal{O}_\theta P(p) \leftrightarrow \mathcal{A}_\theta P(p)$ [T51, Spec]
4. $\mathcal{O}_\theta P(p) \leftrightarrow \mathcal{C}_\theta P(p)$ [Eq, 3 in 2]
5. $\mathcal{O}_\theta P(p) \to d(p)$ [Eq, 4 in 1]
6. $\forall p(\mathcal{O}_\theta P(p) \to d(p))$ [Gen, 5]

$\square$

T52. $\forall p(z(p) \to \neg\mathcal{O}_\theta P(p))$

(For all situations, if a situation is evil, then God is not responsible for such a situation.)

Proof.

1. $z(p)$ [Hip]
2. $\quad \mathcal{O}_\theta P(p) \to d(p)$ [T50, Spec]
3. $\quad \neg d(p) \to \neg\mathcal{O}_\theta P(p)$ [**PC**, 2]
4. $\quad z(p) \to \neg d(p)$ [A6, Spec]
5. $\quad \neg d(p)$ [MP, 1, 4]
6. $\quad \neg\mathcal{O}_\theta P(p)$ [MP, 5, 3]
7. $z(p) \to \neg\mathcal{O}_\theta P(p)$ [DT, 1-6]
8. $\forall p(z(p) \to \neg\mathcal{O}_\theta P(p))$ [Gen, 7]

$\square$

T53. $\forall p((z(p) \wedge \neg\mathcal{S}_\theta P(p)) \to K(p))$

(For all situations, if a situation is evil, and God is not opposed to it, then the situation is contingent.)

Proof.

1. $z(p) \wedge \neg\mathcal{S}_\theta P(p)$ [Hip.]
2. $\quad z(p)$ [**PC**, 1]
3. $\quad \neg\mathcal{S}_\theta P(p)$ [**PC**, 1]
4. $\quad z(p) \to \neg\mathcal{O}_\theta P(p)$ [T52, Spec]
5. $\quad \neg\mathcal{O}_\theta P(p)$ [MP, 2, 4]
6. $\quad \mathcal{O}_\theta P(p) \leftrightarrow \mathcal{A}_\theta P(p)$ [T50, Spec.]
7. $\quad \neg\mathcal{O}_\theta P(p) \leftrightarrow \neg\mathcal{A}_\theta P(p)$ [**PC**, 6]
8. $\quad \neg\mathcal{A}_\theta P(p)$ [MP, 5, 7]
9. $\quad \mathcal{A}_\theta P(p) \leftrightarrow \mathcal{C}_\theta P(p)$ [Def 11, $\alpha(p)/P(p)$]
10. $\quad \neg\mathcal{A}_\theta P(p) \leftrightarrow \neg\mathcal{C}_\theta P(p)$ [**PC**, 9]
11. $\quad \neg\mathcal{C}_\theta P(p)$ [**PC**, 8, 10]
12. $\quad \neg\mathcal{C}_\theta P(p) \wedge \neg\mathcal{S}_\theta P(p)$ [**PC**, 11, 3]

13.	$K(p) \leftrightarrow \neg\mathcal{C}_\theta P(p) \wedge \neg\mathcal{S}_\theta P(p)$	[T45, Spec]
14.	$K(p)$	[Eq, 13 in 12]
15.	$(z(p) \wedge \neg\mathcal{S}_\theta P(p)) \rightarrow K(p)$	[DT, 1–14]
16.	$\forall p((z(p) \wedge \neg\mathcal{S}_\theta P(p)) \rightarrow K(p))$	[Gen, 15]

$\square$

5 Semantics for N1

A model $\mathcal{M}$ for **N1** consists of a quadruple $\langle W, R, D, V\rangle$, in which W is a set of 'worlds', R is a relation on W, D is a domain of 'objects', and V is a function such that, if $\mathcal{P}$ is an n-ary predicate in $\mathcal{L}_{N1}$, then $V(\mathcal{P})$ is a set of n+1-tuples in the form $(u_1, u_2, ..., u_n, w)$, in which $u_1, ..., u_n \in D$, and $w \in W$.

In such model every assignment μ is such that, for each variable p of $\mathcal{L}_{N1}$, $\mu(p) \in D$. Since **N1** has only one constant ('θ'), we fix an element $t \in D$ to be its interpretation, i. e., for every μ, $\mu(\theta) = t$, where t is a fixed element in D.

Every *wff* ϕ has a truth-value (1 or 0) at a world with respect to an assignment μ according to the following conditions:

(a) $V_\mu(\phi(x), w) = 1$ *iff* $(\mu(x), w) \in V(\phi)$, and 0 otherwise;
(b) $V_\mu(\neg\phi, w) = 1$ *iff* $V_\mu(\phi, w) = 0$, and 0 otherwise;
(c) $V_\mu(\phi \rightarrow \psi, w)$ *iff* $V_\mu(\phi, w) = 0$ or $V_\mu(\psi, w) = 1$, and 0 otherwise;
(d) $V_\mu(\phi \vee \psi, w) = 1$ *iff* $V_\mu(\phi, w) = 1$ or $V_\mu(\psi, w) = 1$, and 0 otherwise;
(e) $V_\mu(\phi \wedge \psi, w) = 1$ *iff* $V_\mu(\phi, w) = 1$ and $V_\mu(\psi, w) = 1$, and 0 otherwise;
(f) $V_\mu(\mathcal{C}_\theta\phi, w) = 1$ *iff* $V_\mu(\phi, w') = 1$ for every $w' \in W$ such that wRw', and 0 otherwise;
(g) $V_\mu(\mathcal{W}_\theta\phi, w) = 1$ *iff* $V_\mu(\phi, w) = 1$, and 0 otherwise;
(h) $V_\mu(\forall x\phi(x), w) = 1$ *iff* $V_\mu(\phi(x), w) = 1$ for every $x \in D - \{t\}$, and 0 otherwise.

A *wff* ϕ is valid in $\mathcal{M}$ *iff* $V_\mu(\phi, w) = 1$, for every $w \in W$ and every assignment μ.

In the following, we introduce a particular model for **N1**.

Let $\mathcal{M} = \langle W, R, D, V\rangle$ be an interpretation for **N1**, such that $W = \{w_0, w_1, ..., w_n, ...\}$, where $n \in \mathbb{N}$, $R = W \times W$, $D = \mathbb{Z}$, and V is the union of the following sets:

$V(B) = \{(0, w_n) : n \in \mathbb{N}\}$;
$V(z) = \{(-1, w_n) : n \in \mathbb{N}\}$;
$V(P) = V(B) \cup V(z) \cup \{(2n, w_n) : n \in \mathbb{N}\}$;
$V(d) = V(P) - V(z)$;

$V(n) = \{(3n, w_n) : n \in \mathbb{N}^*\}$;
$V(Op) = \{(i, -i, w_n) : i \in \mathbb{Z}^* \text{ and } n \in \mathbb{N}\}$.

We fix 0 as the interpretation of θ, then for every assignment μ:

$$\mu(\theta) = 0;$$
$$\mu(p) = k, \text{ where } k \in \mathbb{Z}^*.$$

It is possible to show that $\mathcal{M}$ is a model for **N1**, and consequently, the axioms of **N1** are valid in $\mathcal{M}$.

6 Final remarks

We presented here our system **N1**, which deals with the Logical Problem of Evil, based essentially on the system introduced by Edward Nieznański in [11]. We believe that **N1** has a more adequate modal characterization. Our philosophical concern was to investigate the allegation that God's omnipotence implies that every situation which is the case, including evil ones, is willed by God. We showed that actually, given a formalization that could be easily accepted by many theists, it is possible to deduce that the attributes of God are not inconsistent with the existence of evil, and more, that religious determinism (as formalized in the system) is false. We showed also that, assuming the formalization given in **N1**, God is neither the direct cause of, nor responsible for every situation; that there are contingent situations; that evil situations that God does not oppose are contingent; among other results.

As a secondary result, we think that **N1** establishes also a very promising approach in applications of formal systems. That contemporary logic has powerful tools to solve relevant problems is beyond any doubt, but we think that our system is an example of how formal logic can be applied to deal with philosophical problems in the fields of Philosophy of Religion and Analytic Theology. Finally, to elaborate a formal theodicy is just one more way to contribute for the establishment of bridges between the fields of Logic and Religion, and we aim at colaborating even more to that.

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Probability, Intuitionistic Logic and Strong Negation

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Abstract

In this paper I present a sequent calculus for intuitionistic first-order logic with strong negation. The semantic is probabilistic, more precisely, it is based on partial conditional probability functions. Soundness and completeness are proved.

Keywords: Probabilistic interpretation, Intuitionistic logic, Strong negation, Conditional probability

Introduction

The notation $\mathrm{Pr}(A, \Gamma)$ will be used for the probability of A conditional on the set of sentences Γ. The main feature of this semantics is that,
$\mathrm{Pr}(A \supset B, \Gamma) = \mathrm{Pr}(B, \Gamma \cup \{A\})$ i.e. the conditional probability of the intuitionistic conditional is the probability of the consequent conditionalized on the antecedent, when $\mathrm{Pr}(B, \Gamma \cup \{A\})$ is defined.

It is a well-known fact that intuitionistic logic is closed to 3-value logic even if it is not a n-value logic. In fact, intuitionistic logic is not verifunctional. For example, $A \supset B$ is true iff, when we find a proof of A, we find a proof of B, i.e., if we discover that A is true, then we discover that B is true.

In the probabilistic context, the situation is slightly more complex. This is so, because intuitionistic negation is not the adequate negation for probabilistic interpretation. For example, $\mathrm{Pr}(\neg(A \supset B), \Gamma)$ cannot be, in general, be $(1 - \mathrm{Pr}((A \supset B), \Gamma))$.

1 A Sequent Calculus for Intuitionistic Predicate Logic with Strong Negation (SCIPLSN)

In what follows, $\sim$ is the strong negation, F is falsity and $\neg$ is the intuitionistic negation and is not a primitive: $\neg A =_{def} A \supset F$.

Definition 1.1. *Let $Con = \{c_1, \ldots, c_n, \ldots\}$ be the set of constants, $Var = \{x_1, \ldots, x_n, \ldots\}$ the set of variables, $Fon = \{f_0^0, \ldots, f_n^m, \ldots\}$ the set of function letters and $Pre = \{A_0^0, \ldots, A_n^m, \ldots\}$ the set of predicate letters (the superscript indicates the number of arguments).*

The set Ter of terms is defined as:
(i) $Con \cup Var \subseteq Ter$
(ii) If $t_{i_1}, \ldots, t_{i_m} \in Ter$, then $f_n^m(t_{i_1}, \ldots, t_{i_m}) \in Ter$
(iii) Nothing else is in Ter.

The set $WFFs$ of well-formed formulas is defined as:
(i) $\{F, A_n^m(t_{i_1}, \ldots, t_{i_m})\} \subseteq WFF$ for any $i, n, m \in \mathbb{N}$;
(ii) if $A, B \in WFF$, then $\sim A, \forall x_i A, \exists x_i A, (A \wedge B), (A \vee B), (A \supset B) \in WFF$;
(iii) Nothing else is in WFF.

For a wff $\forall x_i A$ (resp. $\exists x_i A$), A is call the *scope* of $\forall x_i$ (resp. $\exists x_i$).

An occurrence of a variable x_i in A which is not in the scope of $\forall x_i$ (resp. $\exists x_i$) nor immediately preceded by $\forall$ (resp. $\exists$) is said to be $free$.

An occurrence of a variable x_i in A which is in the scope of $\forall x_i$ (resp. $\exists x_i$) or immediately preceded by $\forall$ (resp. $\exists$) is said to be *bound*.

Definition 1.2. *Let A be a wff and t a term. $A[t|x_i]$ is the wff obtained by the substitution of all the free occurrences of x_i in A by t.*
t is said to be free for x_i in A iff no variable occurring in t is bound in $A[t|x_i]$.

Axioms

$A, \Gamma \Rightarrow A$ A1

$F, \Gamma \Rightarrow C$ A2

$\Gamma \Rightarrow\ \sim F$ A3

Logical Rules

$$\frac{\Gamma \Rightarrow A}{{\sim}A, \Gamma \Rightarrow C}\ L{\sim}$$

$$\frac{A, \Gamma \Rightarrow C}{{\sim}{\sim}A, \Gamma \Rightarrow C}\ L{\sim}{\sim} \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow {\sim}{\sim}A}\ R{\sim}{\sim}$$

$$\frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}\ L\wedge \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B,}\ R\wedge$$

$$\frac{A, \Gamma \Rightarrow C \qquad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C}\ L\vee \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\ R\vee_1$$

$$\frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}\ R\vee_2$$

$$\frac{{\sim}A, \Gamma \Rightarrow C \qquad {\sim}B, \Gamma \Rightarrow C}{{\sim}(A \wedge B), \Gamma \Rightarrow C}\ L{\sim}\wedge \qquad \frac{\Gamma \Rightarrow\ {\sim}A}{\Gamma \Rightarrow\ {\sim}(A \wedge B)}\ R{\sim}\wedge_1$$

$$\frac{\Gamma \Rightarrow\ {\sim}B}{\Gamma \Rightarrow\ {\sim}(A \wedge B)}\ R{\sim}\wedge_2$$

$$\frac{{\sim}A, {\sim}B, \Gamma \Rightarrow C}{{\sim}(A \vee B), \Gamma \Rightarrow C}\ L{\sim}\vee \qquad \frac{\Gamma \Rightarrow {\sim}A \quad \Gamma \Rightarrow {\sim}B}{\Gamma \Rightarrow {\sim}(A \vee B)}\ R{\sim}\vee$$

$$\frac{\Gamma \Rightarrow A \qquad B, \Gamma \Rightarrow C}{A \supset B, \Gamma \Rightarrow C}\ L \supset \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \supset B}\ R \supset$$

$$\frac{A, {\sim}B, \Gamma \Rightarrow C}{{\sim}(A \supset B), \Gamma \Rightarrow C}\ L{\sim}{\supset} \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow {\sim}B}{\Gamma \Rightarrow {\sim}(A \supset B)}\ R{\sim}{\supset}$$

$$\frac{A[t|x], \Gamma \Rightarrow C}{\forall x A, \Gamma \Rightarrow C}\ L\,\forall \qquad \frac{\Gamma \Rightarrow A[y|x]}{\Gamma \Rightarrow \forall x A}\ R\,\forall$$

$$\frac{A[y|x], \Gamma \Rightarrow C}{\exists x A, \Gamma \Rightarrow C}\ L\,\exists \qquad \frac{\Gamma \Rightarrow A[t|x]}{\Gamma \Rightarrow \exists x A}\ R\,\exists$$

Here we suppose that t is free for x in A and $A[t|x]$ is the wff obtained from A by replacing all the free occurrences of x by t. For $R\ \forall$, we also need the following restriction: y must not be free in Γ, $\forall xA$. For $L\ \exists$, we also need the following restriction: y must not be free in Γ, $\forall xA$ and C.

$$\frac{\sim A[t|x], \Gamma \Rightarrow C}{\sim\exists xA, \Gamma \Rightarrow C}\ L{\sim}\exists \qquad \frac{\Gamma \Rightarrow\ \sim A[y|x]}{\Gamma \Rightarrow\ \sim\exists xA}\ R{\sim}\exists$$

$$\frac{\sim A[y|x], \Gamma \Rightarrow C}{\sim\forall xA, \Gamma \Rightarrow C}\ L{\sim}\forall \qquad \frac{\Gamma \Rightarrow\ \sim A[t|x]}{\Gamma \Rightarrow\ \sim\forall xA}\ R{\sim}\forall$$

For $R\ {\sim}\exists$, we also need the following restriction: y must not be free in Γ, ${\sim}\exists xA$. For $L\ {\sim}\forall$, we also need the following restriction: y must not be free in Γ, ${\sim}\forall xA$ and C.

From a proof-theoretic point of view, $\Gamma \Rightarrow\ \sim A$ can be interpreted as "from Γ we have a constructive proof of the falsity of A". The introduction of strong negation gives us a conservative extension of intuitionistic logic: every derivable sequent of intuitionistic logic is a derivable sequent of intuitionistic logic with strong negation. Moreover, in intuitionistic logic we have $\Gamma \Rightarrow A$ or $\Gamma \not\Rightarrow A$. In intuitionistic logic with strong negation we have $\Gamma \Rightarrow A$ or $\Gamma \Rightarrow\ \sim A$ or ($\Gamma \not\Rightarrow A$ and $\Gamma \not\Rightarrow\ \sim A$.

We will also use the *Cut Rule*

$$\frac{\Gamma \Rightarrow A \qquad A, \Delta \Rightarrow C}{\Delta, \Gamma \Rightarrow C}{\scriptstyle Cut}$$

without proving its admissibility. See Negri and von Plato [19].

2 Partial Conditional Probability Functions

We now characterize the notion of *partial conditional probability function*.

Definition 2.1. *A partial conditional probability function is any partial function*

$$\mathrm{Pr} : WFF \times 2^{WFF} \to [0, 1]$$

which satisfies some postulates that will be specified below.

A background Γ is said to be Pr-abnormal iff, for any A, $\Pr(A, \Gamma) = 1$. Otherwise, it is Pr-normal.

Two partial conditional probability functions that give the same value for the same argument when defined are identical, i.e. they *are* the same function. When $\Pr(A, \Gamma)$ is not defined, we will say that the probability of A is unknown (the interpretation of probability is clearly subjective) giving the background Γ. A first general constraint on partial conditional probability functions is the condition:

Probabilistic Equivalence (PE). Let A and B be two wffs. We will say that A and B are probabilistically equivalent iff, for any Pr and any Γ
(i) $\Pr(A, \Gamma)$ and $\Pr(B, \Gamma)$ are both defined or both undefined;
(ii) $\Pr(A, \Gamma) = \Pr(B, \Gamma)$ when defined.

Probabilistic equivalence is the strongest semantic equivalence relation. See Theorem 2.5 below.

The following definition will be useful.

Definition 2.2. *A n-permutation is a bijection* $per_n : \{1, \ldots, n\} \to \{1, \ldots, n\}$.

We will write $\bigwedge_{i=1}^{n} A_i$ for $(A_1 \wedge (\ldots \wedge A_n) \ldots)$ and $\bigvee_{i=1}^{n} A_i$ for $(A_1 \vee (\ldots \vee A_n) \ldots)$.

We restrict the set of partial probability functions to those which satisfy the following postulates:

DF. 1. *If* $\Pr(A_j, \Gamma) = 0$ *for some* $1 \leq j \leq n$*, then* $\Pr(\bigwedge_{i=1}^{n} A_i, \Gamma) = 0$*;*

DF. 2. *If* $\Pr(A_j, \Gamma) = 1$ *for some* $1 \leq j \leq n$*, then* $\Pr(\bigvee_{i=1}^{n} A_i, \Gamma) = 1$*;*

Postulates DF.1 and DF.2 are the rules governing "unknown".

The following postulates are also satisfied. When the probabilities are known:

POS. 3. $0 \leq \Pr(A, \Gamma) \leq 1$*;*

POS. 4. *If* $A \in \Gamma$*, then* $\Pr(A, \Gamma) = 1$*;*

POS. 5. $\Pr(\bigvee_{i=1}^{n} A_i, \Gamma) = \Pr(A_1, \Gamma) + \Pr(\bigvee_{i=2}^{n} A_i, \Gamma) - \Pr(A_1 \wedge (\bigvee_{i=2}^{n} A_i, \Gamma))$;

POS. 6. $\Pr(\bigwedge_{i=1}^{n} A_i, \Gamma) = \Pr(A_1, \Gamma) \times \Pr(\bigwedge_{i=2}^{n} A_i, \Gamma \cup \{A_1\})$;

POS. 7. $\Pr(\bigwedge_{i=1}^{n} A_i, \Gamma) = \Pr(\bigwedge_{i=1}^{n} A_{per_n(i)}, \Gamma)$

POS. 8. $\Pr(A \supset B, \Gamma) = \Pr(B, \Gamma \cup \{A\})$;

POS. 9. *If* Γ *is Pr-normal, then* $\Pr({\sim}A, \Gamma) =$
(1) $1 - \Pr(A, \Gamma)$ *if* A *is an atom or* F *or* $(B \wedge C)$ *or* $(B \vee C)$ *or* $\forall x A$ *or* $\exists x A$;
(2) $\Pr(B, \Gamma) \times \Pr({\sim}C, \Gamma \cup \{B\})$ *if* A *is* $(B \supset C)$;
(3) $\Pr(B, \Gamma)$ *if* A *is* ${\sim}B$;

POS. 10. $\Pr(C, \Gamma \cup \{\bigwedge_{i=1}^{n} A_i\}) = \Pr(C, \Gamma \cup \{A_1, \ldots, A_n\})$;

POS. 11. *If* Γ *is Pr-normal, then* $\Pr(F, \Gamma) = 0$;

POS. 12. *If, for any* Δ, $\Pr(A, \Gamma \cup \Delta) = 1$,
then for any B *and* C, $\Pr(C, \Gamma \cup \Delta \cup \{B\}) = \Pr(C, \Gamma \cup \Delta \cup \{(A \supset B)\})$

POS. 13. *If* $\Pr(C, \Gamma \cup \{A_i\}) = 1$ *for any* i *such that* $1 \leq i \leq n$, *then* $\Pr(C, \Gamma \cup \{\bigvee_{i=1}^{n} A_i\}) = 1$;

POS. 14. *If* $\Pr(C, \Gamma \cup \{{\sim}A_1, \ldots, {\sim}A_n\}) = 1$, *then* $\Pr(C, \Gamma \cup \{{\sim}(\bigvee_{i=1}^{n} A_i\})) = 1$;

POS. 15. *If* $\Pr(C, \Gamma \cup \{{\sim}A_i\}) = 1$ *for any* i *such that* $1 \leq i \leq n$, *then* $\Pr(C, \Gamma \cup \{{\sim}(\bigwedge_{i=1}^{n} A_i\})) = 1$;

POS. 16. $\Pr(\forall x A, \Gamma) = \lim_{n \to \infty} \Pr(\bigwedge_{i=1}^{n} A[t_i|x], \Gamma)$ *where* $t_1, \ldots, t_n, \ldots$ *is an enumeration of all the terms that are free for* x *in* A ;

POS. 17. $\Pr(\exists x A, \Gamma) = \lim_{n \to \infty} \Pr(\bigvee_{i=1}^{n} A[y_i|x], \Gamma)$ *where* $y_1, \ldots, y_n, \ldots$ *is an enumeration of all the variables which are not free in* A *and* Γ;

POS. 18. *If* $\Pr(C, \Gamma \cup \{A[t|x]\}) = 1$, *then* $\Pr(C, \Gamma \cup \{\forall x A\}) = 1$ *where* t *is free for* x *in* A;

POS. 19. *If* $\Pr(C, \Gamma \cup \{A[y|x]\}) = 1$, *then* $\Pr(C, \Gamma \cup \{\exists x A\}) = 1$ *where* y *is not free in* A, Γ *and* C;

POS. 20. *If* $\Pr(A[y|x], \Gamma) = 1$ *with* y *not free in* Γ *nor in* A *(or* $y = x$*), then* $\Pr(A[t|x], \Gamma) = 1$ *where* t *is free for* x *in* A.

Remarks 2.3.

• *DF.1-DF.2 are quite intuitive. For example, if* $\Pr(A, \Gamma) = 0$*, then* $\Pr(B \wedge A, \Gamma) = 0$*,* $\Pr(B, \Gamma)$ *being defined or not.*

• *Unknown is not a value but a lack of value. So, arithmetical operations cannot be applied to expressions with unknown values, even equality. Even if both* $\Pr(A, \Gamma)$ *and* $\Pr(B, \Delta)$ *are both unknown, this doesn't mean that they are equal.The only legitimate uses of expressions that have unknown values are those explicitly given by DF.1-DF.2. So, if* $\Pr(p, \Gamma)$ *is unknown, POS.6 does not hold, i.e.* $\Pr(p \wedge {\sim}p, \Gamma)$ *is undefined if* Γ *is Pr-normal.*

One can easily prove that

Theorem 2.4. $\Pr(\bigvee_{i=1}^{n} A_i, \Gamma) = \Pr(\bigvee_{i=1}^{n} A_{per_n(i)}, \Gamma)$.

Proof. We just give a sketch of the proof. We proceed by induction using POS.5, POS.6 and POS.7. It is clear and that any permutation of the disjuncts preserves the value or lack of value. □

Furthermore, using POS.10 together with POS.5 and POS.6, we can easily prove that, when defined, for any $n \in \mathbb{N}$, $\Pr(\bigwedge_{i=1}^{n} A_i, \Gamma) \leq \Pr(\bigwedge_{i=1}^{n-1} A_i, \Gamma)$ and $\Pr(\bigvee_{i=1}^{n} A_i, \Gamma) \geq \Pr(\bigvee_{i=1}^{n-1} A_i, \Gamma)$. So the sequences $\Pr(A_1, \Gamma), \ldots, \Pr(\bigwedge_{i=1}^{n} A_i, \Gamma), \ldots,$ and $\Pr(A_1, \Gamma), \ldots, \Pr(\bigvee_{i=1}^{n} A_i, \Gamma), \ldots,$ are respectively decreasing and increasing. As these sequences are bounded (by POS.3), it follows, by an elementary result of real numbers analysis, that their limits exist and are in $[0, 1]$. This insures that POS.16 and POS.17 are not only very intuitive constraints but are adequate.

POS.8, $\Pr(A \supset B, \Gamma) = \Pr(B, \Gamma \cup \{A\})$ calls for some comments. It is simply the expression of the very intuitive interpretation of the probability of the conditional as the conditional probability. David Lewis showed that this cannot apply to material conditional, i.e. to ${\sim}A \vee B$. But it can be applied to $A \supset B$ when " $\supset$ " is the probabilistic conditional: The probability of $A \supset B$ given the background Γ is just the probability of B when A is hypothetically add to Γ. Lewis' proof does not hold in intuitionistic logic nor in intuitionistic logic with strong negation. See [9].

The following theorem will be useful.

Theorem 2.5. *(Substituability of Probabilistic Equivalents in Background) Let A and B be two wffs and Γ a set of wffs. If for any Δ, $\Pr(A, \Gamma \cup \Delta)$ is known iff $\Pr(B, \Gamma \cup \Delta)$ is known and $\Pr(A, \Gamma \cup \Delta) = \Pr(B, \Gamma \cup \Delta)$ when both are known, then, for any C, $\Pr(C, \Gamma \cup \Delta \cup \{A\}) = \Pr(C, \Gamma \cup \Delta \cup \{B\})$ when both are known.*

Proof.

$\Pr(C \wedge A, \Gamma \cup \Delta) =$	
$\quad \Pr(C, \Gamma \cup \Delta) \times \Pr(A, \Gamma \cup \Delta \cup \{C\})$	POS.13
$\Pr(C \wedge B, \Gamma \cup \Delta) =$	
$\quad \Pr(C, \Gamma \cup \Delta) \times \Pr(B, \Gamma \cup \Delta \cup \{C\})$	POS.13
$\Pr(C \wedge A, \Gamma \cup \Delta) = \Pr(C \wedge B, \Gamma \cup \Delta)$	Assumption + algebra
$\Pr(A \wedge C, \Gamma \cup \Delta) = \Pr(B \wedge C, \Gamma \cup \Delta)$	POS.15
$\Pr(A, \Gamma \cup \Delta) \times \Pr(C, \Gamma \cup \Delta \cup \{A\}) =$	
$\Pr(B, \Gamma \cup \Delta) \times \Pr(C, \Gamma \cup \Delta \cup \{B\})$	POS.13
$\Pr(C, \Gamma \cup \Delta \cup \{A\}) = \Pr(C, \Gamma \cup \Delta \cup \{B\})$	Assumption + algebra

$\square$

Theorem 2.6. *$\sim(A \supset B)$ and $(A \wedge \sim B)$ are substitutable in backgrounds.*

Proof. Trivial by POS.9 (2).

$\square$

The following definition will be useful.

Definition 2.7. *Let $PB \subseteq WFF$ be the set of wffs that are not of the form $A \supset B$ (PB for pseudo boolean).*

Theorem 2.8. *For any wff $A \in PB$, if Γ is Pr-normal, then $\Pr(\sim A, \Gamma) = 1 - \Pr(A, \Gamma)$ when $\Pr(A, \Gamma)$ is defined.*

Proof. This is a trivial consequence of POS.9 (1) and when A is $\sim B$, the conclusion follows from induction using POS.9 (3). $\square$

This is the "classical" case. Consider the case where $A \notin PB$. We have $\Pr(\sim(B \supset C), \Gamma) = \Pr(B, \Gamma) \times \Pr(\sim C, \Gamma \cup \{B\})$, when defined. The easy case is when $C \in PB$: $\Pr(\sim(B \supset C), \Gamma) = \Pr(B, \Gamma) \times (1 - \Pr(C, \Gamma \cup \{B\})) = \Pr(B, \Gamma) \times (1 - \Pr(B \supset C), \Gamma)$. We have three sub-cases:

(1) If $\Pr(B \supset C, \Gamma) = 0$, then $\Pr(\sim(B \supset C), \Gamma) = \Pr(B, \Gamma)$.

(2) If $0 < \Pr((B \supset C), \Gamma) < 1$, then $0 < \Pr(\sim(B \supset C), \Gamma) < \Pr(B, \Gamma)$

(3) If $\Pr(B \supset C), \Gamma) = 1$, then $\Pr(\sim(B \supset C), \Gamma) = 0$.

(1) If $\Pr((B \supset C), \Gamma) = 0$, then $\Pr(C, \Gamma \cup \{B\}) = 0$. By hypothesis, $C \in PB$ and thus $\Pr(\sim C, \Gamma \cup \{B\})) = (1 - \Pr(C, \Gamma \cup \{B\})) = 1$ As $\Pr(\sim(B \supset C), \Gamma) = \Pr(B, \Gamma) \times \Pr(\sim C, \Gamma \cup \{B\}))$ we have $\Pr(\sim(B \supset C), \Gamma) = Pr(B, \Gamma) \times \Pr(\sim C, \Gamma \cup \{B\}) = Pr(B, \Gamma) \times (1 - 0) = Pr(B, \Gamma)$.

(2) We have $0 < \Pr((B \supset C), \Gamma) < 1$ and thus $0 < \Pr(C, \Gamma \cup \{B\}) < 1$
$\Pr(\sim(B \supset C), \Gamma) = \Pr(B, \Gamma) \times \Pr(\sim C, \Gamma \cup \{B\})$.
This implies that $0 < \Pr(\sim(B \supset C), \Gamma) < \Pr(B, \Gamma)$ because $\Pr(\sim C, \Gamma \cup \{B\}) = 1 - \Pr(C, \Gamma \cup \{B\})$ and $0 < 1 - \Pr(C, \Gamma \cup \{B\}) < 1$

(3) We have $\Pr((B \supset C), \Gamma) = 1 = \Pr(C, \Gamma \cup \{B\})$
But $\Pr(\sim(B \supset C), \Gamma) = \Pr(B, \Gamma) \times \Pr(\sim C, \Gamma \cup \{B\}) = \Pr(B, \Gamma) \times (1 - \Pr(C, \Gamma \cup \{B\})) = \Pr(B, \Gamma) \times (1 - 1) = 0$.

We now have to take a closer look to the general case. The problem is with C:
$\Pr((B \supset C), \Gamma) = \Pr(B, \Gamma) \times \Pr(\sim C, \Gamma \cup \{B\})$. If $C \notin PB$ i.e. C is $D \supset E$, we are back to square one. We clearly need a proof based on the number of "$\supset$".

Theorem 2.9. *Let A be $(C_0 \supset (C_1 \supset (C_2 \supset (\ldots \supset (C_{n-1} \supset C_n) \ldots))))$ with $C_n \in PB$ (any A has this form, for some $n \geq 0$). Then, when* $\Pr(\sim A, \Gamma)$ *is defined*

(1) If $\Pr(A, \Gamma) = 0$ *then* $\Pr(\sim A, \Gamma) = \Pr(C_0, \Gamma) \times \Pr(C_1, \Gamma \cup \{C_0\}) \times \Pr(C_2, \Gamma \cup \{C_0, C_1\}) \times \ldots \times \Pr(\sim C_n, \Gamma \cup \{C_0, C_1, C_2 \ldots C_{n-1}\})$*;*

(2) $0 < \Pr(A, \Gamma) < 1$*, then*
$0 < \Pr(\sim A, \Gamma) < \Pr(C_0, \Gamma) \times \Pr(C_1, \Gamma \cup \{C_0\}) \times \Pr(C_2, \Gamma \cup \{C_0, C_1\}) \times \ldots \times \Pr(\sim C_n, \Gamma \cup \{C_0, C_1, C_2 \ldots C_{n-1}\})$*;*

(3) If $\Pr(A, \Gamma) = 1$ *then* $\Pr(\sim A, \Gamma) = 0$

Proof. First of all, by applying POS.8 n times, we have $\Pr(C_0 \supset (C_1 \supset (C_2 \supset$

$(\ldots \supset (C_{n-1} \supset C_n)\ldots))),\Gamma) = \Pr(C_n, \Gamma \cup \{C_0, C_1, C_2 \ldots C_{n-1}\})$

By applying POS.9 (2) n times, we have
$\Pr(\sim(C_0 \supset (C_1 \supset (C_2 \supset (\ldots \supset (C_{n-1} \supset C_n)\ldots)))),\Gamma) = \Pr(C_0,\Gamma) \times \Pr(C_1,\Gamma\cup\{C_0\})\times\Pr(C_2,\Gamma\cup\{C_0,C_1\})\times\ldots\times\Pr(\sim C_n,\Gamma\cup\{C_0,C_1,C_2\ldots C_{n-1}\})$

(1) We need to make sure that: If $\Pr(C_0 \supset (C_1 \supset (C_2 \supset (\ldots \supset (C_{n-1} \supset C_n)\ldots))))),\Gamma) = 0$, then $\Pr(\sim(C_0 \supset (C_1 \supset (C_2 \supset (\ldots \supset (C_{n-1} \supset C_n)\ldots))))),\Gamma) = \Pr(C_0,\Gamma)\times\Pr(C_1,\Gamma\cup\{C_0\})\times\Pr(C_2,\Gamma\cup\{C_0,C_1\})\times\ldots\times\Pr(C_n,\Gamma\cup\{C_0,C_1,C_2\ldots C_{n-1}\})$
i.e. if $\Pr(C_n,\Gamma\cup\{C_0,C_1,C_2\ldots C_{n-1}\}) = 0$, then $\Pr(\sim C_n,\Gamma\cup\{C_0,C_1,C_2\ldots C_{n-1}\})) = 1$.
This is trivial because $C_n \notin PB$ and thus, $\Pr(\sim C_n,\Gamma\cup\{C_0,C_1,C_2\ldots C_{n-1}\}) = (1-\Pr(C_n,\Gamma\cup\{C_0,C_1,C_2\ldots C_{n-1}\})) = (1-0) = 1$.

(2) and (3) are also quite trivial along the same lines. □

Corollary 2.10. *Let A be $(C_0 \supset (C_1 \supset (C_2 \supset (\ldots \supset (C_{n-1} \supset C_n)\ldots))))$ with $C_n \in PB$. Then, when defined,* $\Pr(\sim A,\Gamma) \leq 1 - \Pr(A,\Gamma)$.

Proof. Trivial. □

The following theorems will be useful.

Theorem 2.11. *If* $\Pr(A,\Gamma) = 1$ *and* $\Pr(B,\Gamma)$ *is defined, then* $\Pr(B,\Gamma) = \Pr(B,\Gamma\cup\{A\})$.

Proof.

$\Pr(A \vee B),\Gamma) = 1$	DF.2
$= \Pr(A,\Gamma) + \Pr(B,\Gamma) - \Pr(A \wedge B),\Gamma)$	POS.5
$= 1 + \Pr(B,\Gamma) - \Pr(A \wedge B),\Gamma)$	$\Pr(A,\Gamma) = 1$
$\Pr(B,\Gamma) = \Pr(A \wedge B,\Gamma)$	Algebra
$\Pr(B,\Gamma) = \Pr(A,\Gamma) \times \Pr(B,\Gamma\cup\{A\})$	POS.6
$\Pr(B,\Gamma) = \Pr(B,\Gamma\cup\{A\})$	$\Pr(A,\Gamma) = 1$

□

Theorem 2.12. *If* $\Pr(\sim A,\Gamma) = 1$*, then* $\Pr(A,\Gamma) = 0$

Proof. If $A \in PB$, it is a trivial consequence of POS.9 (1)-(3).
If not, A is $(C_0 \supset (C_1 \supset (C_2 \supset (\ldots \supset (C_{n-1} \supset C_n)\ldots))))$ with $C_n \in PB$.

$\Pr({\sim}(C_0 \supset (C_1 \supset (C_2 \supset (\dots$	
$\supset (C_{n-1} \supset C_n)\dots)))), \Gamma) = 1$	Hypothesis
$= \Pr(C_0, \Gamma) \times \Pr(C_1, \Gamma \cup \{ C_0\}) \times \dots \times$	
$\Pr({\sim}(C_n, \Gamma \cup \{C_0, \dots, C_{n-1}\})$	POS.9 (2) n times
$\Pr(C_0, \Gamma) = \Pr(C_1, \Gamma \cup \{C_0\}) = \dots =$	
$\Pr(C_{n-1}, \Gamma \cup \{C_0, \dots, C_{n-2}\}) = 1$	Algebra
$\Pr(C_0, \Gamma) = \Pr(C_1, \Gamma) = \dots =$	
$\Pr(C_{n-1}, \Gamma) = 1$	Thm 2.11 $(n-1)$ times
$\Pr({\sim}C_n, \Gamma \cup \{C_0, \dots, C_{n-1}\}) = 1$	Algebra
$\Pr(C_n, \Gamma \cup \{C_0, \dots, C_{n-1}\}) = 0$	$C_n \in PB$
$\Pr(C_n, \Gamma) = 0$	$\Pr(C_i, \Gamma) = 1$ for $0 \leq i \leq (n-1)$ and Thm 2.11 $(n-1)$ times

$\Pr(C_0 \supset (C_1 \supset (C_2 \supset (\dots$	
$\supset (C_{n-1} \supset C_n)\dots)))), \Gamma) =$	
$\Pr(C_n, \Gamma \cup \{C_0, \dots, C_{n-1}\}) =$	
$\Pr(C_n, \Gamma) = 0$	$\Pr(C_i, \Gamma) = 1$ for $0 \leq i \leq (n-1)$ and Thm 2.11 $(n-1)$ times

□

The above theorems show that the probabilistic semantic value of formulas containing strong negation of "horseshoe" is far from being trivial.

Theorem 2.13. *If $A_1, A_2 \in PB$, then ${\sim}(A_1 \wedge A_2)$ and $({\sim}A_1 \vee {\sim}A_2)$ are probabilisticaly equivalent.*

Proof. We have to prove that, for any Γ, $\Pr({\sim}(A_1 \wedge A_2), \Gamma) = \Pr(({\sim}A_1 \vee {\sim}A_2), \Gamma)$ or both are unknown. Let us suppose that $\Pr({\sim}(A_1 \wedge A_2), \Gamma)$ is known. $\Pr({\sim}(A_1 \wedge A_2), \Gamma) = 1 - \Pr((A_1 \wedge A_2), \Gamma)$ by POS.9 (1).

As the general proof uses Bayes' theorem, we need to consider a special case. Let us suppose that $\Pr({\sim}A_1, \Gamma) = 0$ (a similar proof holds for A_2).

(α)

$\Pr({\sim}(A_1 \wedge A_2), \Gamma) = 1 - \Pr((A_1 \wedge A_2), \Gamma)$	POS.9 (1)
$\Pr({\sim}A_1, \Gamma) = 0$	Hypothesis
$\Pr(A_1, \Gamma) = 1$	$A_1 \in PB$
$\Pr((A_1 \wedge A_2), \Gamma) = \Pr(A_2, \Gamma)$	DF.1
$1 - \Pr((A_1 \wedge A_2), \Gamma) = 1 - \Pr(A_2, \Gamma)$	Algebra
$\Pr({\sim}(A_1 \wedge A_2), \Gamma) = \Pr(({\sim}A_1 \vee {\sim}A_2), \Gamma)$	DF.2

(β) Let us suppose that $\Pr({\sim}A_1, \Gamma) \neq 0$ and $\Pr({\sim}A_2, \Gamma) \neq 0$.

In that case, $\Pr({\sim}(A_1 \wedge A_2), \Gamma)$ and $\Pr(({\sim}A_1 \vee {\sim}A_2), \Gamma)$ are both undefined if and only if one of them is undefined.

We prove: $\Pr({\sim}(A_1 \wedge A_2), \Gamma) = \Pr(({\sim}A_1 \vee {\sim}A_2), \Gamma)$.

$$\begin{array}{ll}
\Pr({\sim}(A_1 \wedge A_2), \Gamma) = 1 - \Pr((A_1 \wedge A_2), \Gamma) & \text{POS.9 (1)} \\
= 1 - \Pr(A_1, \Gamma \cup \{A_2\})) \times \Pr(A_2, \Gamma) & \text{POS.6} \\
= \Pr({\sim}A_2, \Gamma) + \Pr(A_2, \Gamma) - \Pr(A_1, \Gamma \cup \{A_2\})) & \\
\quad \times \Pr(A_2, \Gamma) & A_2 \in PB \\
= \Pr({\sim}A_2, \Gamma) + \Pr(A_2, \Gamma) \times (\Pr({\sim}A_1, \Gamma \cup \{A_2\})) & \text{Algebra} \\
= \Pr({\sim}A_1, \Gamma) - \Pr({\sim}A_1, \Gamma) + \Pr({\sim}A_2, \Gamma) & \\
\quad + \Pr(A_2, \Gamma) \times (\Pr({\sim}A_1, \Gamma \cup \{A_2\})) & \text{Algebra} \\
= \Pr({\sim}A_1, \Gamma) + \Pr({\sim}A_2, \Gamma) - \Pr({\sim}A_1, \Gamma) & \\
\quad + \Pr(A_2, \Gamma) \times (\Pr({\sim}A_1, \Gamma \cup \{A_2\})) & \text{Algebra} \\
= \Pr({\sim}A_1, \Gamma) + \Pr({\sim}A_2, \Gamma) - \Pr({\sim}A_1, \Gamma) \times & \\
\quad \left(1 - \dfrac{\Pr(A_2, \Gamma) \times (\Pr({\sim}A_1, \Gamma \cup \{A_2\}))}{\Pr({\sim}A_1, \Gamma)}\right) & \text{Algebra} \\
= \Pr({\sim}A_1, \Gamma) + \Pr({\sim}A_2, \Gamma) - & \\
\quad \Pr({\sim}A_1, \Gamma) \times (1 - \Pr A_2, \Gamma \cup \{{\sim}A_1\}) & \text{Bayes} \\
= \Pr({\sim}A_1, \Gamma) + \Pr({\sim}A_2, \Gamma) - & \\
\quad \Pr({\sim}A_1, \Gamma) \times (\Pr({\sim}A_2), \Gamma \cup \{{\sim}A_1\}) & A_2 \in PB \\
= \Pr({\sim}A_1, \Gamma) + \Pr({\sim}A_2, \Gamma) - \Pr(({\sim}A_1 \wedge {\sim}A_1), \Gamma) & \text{POS.6} \\
= \Pr(({\sim}A_1 \vee {\sim}A_2), \Gamma) & \text{POS.5}
\end{array}$$

□

Similarly, we have:

$\Pr({\sim}\exists x_i A, \Gamma) = \Pr(\forall x_i {\sim}A, \Gamma)$

$$\begin{array}{ll}
\Pr({\sim}\exists x_i A, \Gamma) = 1 - \Pr(\exists x_i A, \Gamma) & \text{POS.9 (1)} \\
= 1 - \lim\limits_{n \to \infty} \Pr(\bigvee\limits_{i=1}^{n} A[t_{i_n}|x], \Gamma) & \text{POS.17} \\
= \lim\limits_{n \to \infty} (1 - \Pr(\bigvee\limits_{i=1}^{n} A[t_{i_n}|x], \Gamma)) & \text{Elementary calculus} \\
= \lim\limits_{n \to \infty} \Pr({\sim} \bigvee\limits_{i=1}^{n} A[t_{i_n}|x], \Gamma) & \text{POS.9 (1)}
\end{array}$$

In what follows, we will use "unknown", "unknown value" and "undefined" for the same purpose depending on the context.

$\Gamma \Rightarrow A$ is a valid sequent according to partial probabilistic interpretations iff, for any Pr satisfying all the DF and all the POS,

$$\Pr(A, \Gamma \cup \Delta) = 1$$

for all Δ. This intuition is very robust. A is a valid consequence of Γ iff, the probability that A is 1 and remains invariant, regardless of what is added to the background Γ.

Theorem 2.14. *If* Γ *is Pr-abnormal, then* $\Gamma \cup \{A\}$ *is Pr-abnormal.*

Proof.

$\Pr(A, \Gamma) = \Pr(B, \Gamma) = 1$	Γ is Pr-abnormal
$\Pr(B, \Gamma) = \Pr(B, \Gamma \cup \{A\}) = 1$	Theorem 2.11

□

Theorem 2.15. *If* $\Pr(A \wedge {\sim}A, \Gamma) = 1$*, then* Γ *is Pr-abnormal.*

Proof. Let us suppose that Γ is Pr-normal.

$\Pr(A \wedge {\sim}A, \Gamma) = 1$	Hypothesis
$\Pr(A, \Gamma) \times \Pr({\sim}A, \Gamma \cup \{A\}) = 1$	POS.6
$\Pr(A, \Gamma) = \Pr({\sim}A, \Gamma \cup \{A\}) = 1$	Algebra
$\Pr({\sim}A, \Gamma) = 1$	Theorem 2.11
$\Pr({\sim}{\sim}A, \Gamma) = 0$	Theorem 2.12
$\Pr(A, \Gamma) = 0$	POS. 9 (3)
$1 = 0$	

Thus Γ is Pr-abnormal. □

Theorem 2.16. *If* Γ *is Pr-normal but* $\Gamma \cup \{A\}$ *is Pr-abnormal and* $\Pr(A, \Gamma)$ *is defined, then* $\Pr(A, \Gamma) = 0$.

Proof.

$\Pr(F, \Gamma \cup \{A\}) = 1$	$\Gamma \cup \{A\}$ is Pr-abnormal
$\Pr(A, \Gamma) \neq 0$	Hypothesis
$\Pr(F \wedge A, \Gamma) = \Pr(A \wedge F, \Gamma)$	POS.7
$\Pr(A \wedge F, \Gamma) = \Pr(A, \Gamma) \times \Pr(F, \Gamma \cup \{A\}$	POS.6
$\Pr(A \wedge F, \Gamma) = \Pr(A, \Gamma)$	$\Gamma \cup \{A\}$ is Pr-abnormal

But this is impossible because, by POS.11 and DF.1 $\Pr(F \wedge A, \Gamma) = 0$. Thus $\Pr(A, \Gamma) = 0$.

□

Theorem 2.17. *If* $\Pr(A, \Gamma) = 1$ *and* $\Pr(A \supset B, \Gamma) = 0$*, then* $\Pr(B, \Gamma) = 0$.

Proof.

$0 = \Pr(A \supset B, \Gamma)$	Hypothesis
$= \Pr(B, \Gamma \cup \{A\})$	POS.8
$= \Pr(B, \Gamma)$	Theorem 2.11

□

Theorem 2.18. *If* $\Pr(A, \Gamma) \neq 0$ *or* $(\Pr(B, \Gamma) \neq 0)$*, then* $\Pr(A \vee B, \Gamma) \neq 0$.

Proof. We proceed by contraposition.

$0 = \Pr(A \vee B, \Gamma)$	Hypothesis
$= \Pr(A, \Gamma) + \Pr(B, \Gamma) - \Pr(A \wedge B, \Gamma)$	POS.5
$= \Pr(A, \Gamma) + \Pr(B, \Gamma) - \Pr(A, \Gamma) \times \Pr(B, \Gamma \cup \{A\})$	POS.6
$= \Pr(A, \Gamma) \times (1 - \Pr(B, \Gamma \cup \{A\})) + \Pr(B, \Gamma)$	Algebra
$\Pr(A, \Gamma) \times (1 - \Pr(B, \Gamma \cup \{A\})) = 0$ and $\Pr(B, \Gamma) = 0$	POS.3
$\Pr(A, \Gamma) = 0$	Algebra

□

3 Soundness

Let us recall the definition of validity: $\Gamma \Rightarrow A$ is a valid sequent according to partial probabilistic interpretations iff, for any Pr satisfying DF.1-DF.2 and POS.3-POS.20,

$$\Pr(A, \Gamma \cup \Delta) = 1$$

for all Δ. We write $\Gamma \mid\mid\!\!- A$.

We have two types of rules:

$$\frac{\Gamma \Rightarrow A}{\Delta \Rightarrow C} \text{ and } \frac{\Gamma \Rightarrow A \qquad \Lambda \Rightarrow B}{\Delta \Rightarrow C}.$$

The former is sound iff $\Gamma \mid\mid\!\!- A$ implies $\Delta \mid\mid\!\!- C$.

The latter is sound iff $\Gamma \mid\mid\!\!- A$ and $\Lambda \mid\mid\!\!- B$ implies $\Delta \mid\mid\!\!- C$.

Theorem 3.1. *The sequent calculus SCILSN is sound according to partial probabilistic interpretations.*

We need to verify the validity of the axioms and the soundness of the rules.

Axioms

A1 $\quad A, \Gamma \Rightarrow A$ is valid.

Proof. By POS.4, for any A, Γ and Pr, $\Pr(A, \Gamma \cup \{A\} \cup \Delta) = 1$

□

A2 $\quad F, \Gamma \Rightarrow A$ is valid.

Proof. We show that $\Gamma \cup \{F\} \cup \Delta$ is Pr-abnormal for any Γ, Pr and Δ.

$\Pr(F, \Gamma \cup \{F\} \cup \Delta) = 1$	POS.4
$\Gamma \cup \{F\} \cup \Delta$ is Pr-abnormal	POS.11
$\Pr(A, \Gamma \cup \{F\} \cup \Delta = 1$	Theorem 2.14
$F, \Gamma \Rightarrow A$ is valid	Definition of validity.

□

A3 $\quad \Gamma \Rightarrow \sim F$ is valid.

Proof. If Γ is Pr-abnormal, il is trivial. If not

$\Pr(\sim F, \Gamma \cup \Delta) = (1 - \Pr(F, \Gamma \cup \Delta))$	POS.9
$\Pr(F, \Gamma \cup \Delta) = 0$	POS.11
$\Pr(\sim F, \Gamma \cup \Delta) = 1$	Algebra
$\Gamma \Rightarrow \sim F$ is valid	Definition of validity.

□

Logical rules

$$\frac{\Gamma \Rightarrow A}{\sim A, \Gamma \Rightarrow C}\ L\sim \text{ is sound.}$$

Proof. If Γ is Pr-abnormal, then by the Theorem 2.14 $\Gamma \cup \{\sim A\}$ is Pr-abnormal and we are done. Else, let us suppose that Γ is Pr-normal. For any Δ,

$\Gamma \Rightarrow A$	Hypothesis
$\Pr(A, \Gamma \cup \Delta) = 1$	Definition of validity
$\Pr(A, \Gamma \cup \Delta \cup \{\sim A\}) = 1$	with $\Delta' = \Delta \cup \{\sim A\}$
$\Pr(\sim A, \Gamma \cup \Delta \cup \{\sim A\}) = 1$	POS.4
$\Pr((\sim A \ \wedge A, \Gamma \cup \Delta \cup \{\sim A\}) =$	
$\quad \Pr((A \ \wedge \sim A, \Gamma \cup \Delta \cup \{\sim A\})$	POS.7

$\Pr((\sim A \wedge A, \Gamma \cup \Delta \cup \{\sim A\}) =$ $\Pr((\sim A, \Gamma \cup \Delta \cup \{\sim A\})$ $\times \Pr((A, \Gamma \cup \Delta \cup \{\sim A\} \cup \{\sim A\})$	POS.6
$\Pr((A \wedge \sim A, \Gamma \cup \Delta \cup \{\sim A\}) = 1$	Algebra
$\Gamma \cup \Delta \cup \{\sim A\}$ is Pr-abnormal	Theorem 2.15
$\Gamma \cup \{\sim A\} \cup \{C\}$ is Pr-abnormal	Theorem 2.14
$\Pr(C, \Gamma \cup \Delta \cup \{\sim A\}) = 1$	Definition of abnormality
$\sim A, \Gamma \Rightarrow C$	Definition of validity

□

$$\frac{A, \Gamma \Rightarrow C}{\sim\sim A, \Gamma \Rightarrow C}\ L{\sim\sim}$$

Proof. For any Δ,

$A, \Gamma \Rightarrow C$	Hypothesis
$\Pr(C, \Gamma \cup \{A\} \cup \Delta) = 1$	Definition of validity
$\Pr(C, \Gamma \cup \{\sim\sim A\} \cup \Delta) = 1$	POS.9 (3) and Theorem 2.5
$\sim\sim A, \Gamma \Rightarrow C$	Definition of validity

□

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow\ \sim\sim A}\ R{\sim\sim}$$

Proof. For any Δ,

$\Gamma \Rightarrow A$	Hypothesis
$\Pr(A, \Gamma \cup \Delta) = 1$	Definition of validity
$\Pr(\sim\sim A, \Gamma \cup \Delta) = 1$	POS.9 (3)
$\Gamma \Rightarrow\ \sim\sim A$	Definition of validity

□

$$\frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}\ L\wedge$$

Proof. For any Δ,

$A, B, \Gamma \Rightarrow C$	Hypothesis
$\Pr(C, \Gamma \cup \{A, B\} \cup \Delta) = 1$	Definition of validity
$\Pr(C, \Gamma \cup \{A \wedge B\} \cup \Delta) = 1$	POS.10
$A \wedge B, \Gamma \Rightarrow C$	Definition of validity

□

$$\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} R\wedge$$

Proof. For any Δ,

$\Gamma \Rightarrow A$	Hypothesis
$\Gamma \Rightarrow B$	Hypothesis
$\Pr(A, \Gamma \cup \Delta) = 1$	Definition of validity
$\Pr(B, \Gamma \cup \Delta) = 1$	Definition of validity
$\Pr(B, \Gamma \cup \Delta \cup \{A\}) = 1$	Theorem 2.11
$\Pr(A \wedge B, \Gamma \cup \Delta) =$ $\Pr(A, \Gamma \cup \Delta) \times \Pr(B, \Gamma \cup \Delta \cup \{A\})$	POS.6
$\Pr(A \wedge B, \Gamma \cup \Delta) = 1 \times 1 = 1$	Algebra
$\Gamma \Rightarrow A \wedge B$	Definition of validity

□

$$\frac{A, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C} L\vee$$

Proof. For any Δ,

$A, \Gamma \Rightarrow C$	Hypothesis
$B, \Gamma \Rightarrow C$	Hypothesis
$\Pr(C, \Gamma \cup \{A\} \cup \Delta) = 1$	Definition of validity
$\Pr(C, \Gamma \cup \{B\} \cup \Delta) = 1$	Definition of validity
$\Pr(C, \Gamma \cup \{A \vee B\} \cup \Delta) = 1$	POS.13
$A \vee B, \Gamma \Rightarrow C$	Definition of validity

□

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B} R\vee_1$$

Proof. For any Δ,

$\Gamma \Rightarrow A$	Hypothesis
$\Pr(A, \Gamma \cup \Delta) = 1$	Definition of validity
$\Pr(A \vee B, \Gamma \cup \Delta) = 1$	DF.2
$\Gamma \Rightarrow A \vee B$	Definition of validity

□

$$\frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B} R\vee_2$$

Proof. Trivial by the soundness of $R\vee_1$ and Theorem 2.4

□

$$\frac{\sim A, \Gamma \Rightarrow C \quad \sim B, \Gamma \Rightarrow C}{\sim(A \wedge B), \Gamma \Rightarrow C}\ L{\sim}\wedge$$

Proof. For any Δ,

$\sim A, \Gamma \Rightarrow C$	Hypothesis
$\sim B, \Gamma \Rightarrow C$	
	Hypothesis
$\Pr(C, \Gamma \cup \{\sim A\} \cup \Delta) = 1$	Definition of validity
$\Pr(C, \Gamma \cup \{\sim B\} \cup \Delta) = 1$	Definition of validity
$\Pr(C, \Gamma \cup \{\sim(A \wedge B)\} \cup \Delta) = 1$	POS.15
$\sim(A \wedge B), \Gamma \Rightarrow C$	Definition of validity

□

$$\frac{\Gamma \Rightarrow\ \sim A}{\Gamma \Rightarrow \sim(A \wedge B)}\ R{\sim}\wedge_1$$

Proof. For any Δ,

$\Gamma \Rightarrow\ \sim A$	Hypothesis
$\Pr(\sim A, \Gamma \cup \Delta) = 1$	Definition of validity
$\Pr(A, \Gamma \cup \Delta) = 0$	Theorem 2.12
$\Pr(\sim(A \wedge B), \Gamma \cup \Delta) =$	
$\quad 1 - \Pr((A \wedge B), \Gamma \cup \Delta)$	POS.9 (1)
$\Pr((A \wedge B), \Gamma \cup \Delta) =$	
$\quad \Pr(A, \Gamma \cup \Delta) \times \Pr(B, \Gamma \cup \Delta \cup \{A\})$	POS.6
$\Pr((A \wedge B), \Gamma \cup \Delta) = 0$	Algebra
$\Pr(\sim(A \wedge B), \Gamma \cup \Delta) = 1 - 0 = 1$	Algebra
$\Gamma \Rightarrow \sim(A \wedge B)$	Definition of validity

□

$$\frac{\Gamma \Rightarrow\ \sim A}{\Gamma \Rightarrow \sim(A \wedge B)}\ R{\sim}\wedge_2$$

Proof. Trivial from the soundness of $R{\sim}\wedge_1$ and POS.7.

□

$$\frac{\sim A, \sim B, \Gamma \Rightarrow C}{\sim(A \vee B), \Gamma \Rightarrow C}\ L{\sim}\vee$$

Proof. For any Δ

$\sim A, \sim B, \Gamma \Rightarrow C$	Hypothesis
$\Pr(C, \Gamma \cup \Delta \cup \{\sim A, \sim B\}) = 1$	Definition of validity
$\Pr(C, \Gamma \cup \Delta \cup \{\sim (A \vee B)\}) = 1$	POS.14
$\sim(A \vee B), \Gamma \Rightarrow C$	Definition of validity

$\square$

$$\frac{\Gamma \Rightarrow\ \sim A \quad \Gamma \Rightarrow\ \sim B}{\Gamma \Rightarrow\ \sim(A \vee B)}\ R{\sim}\vee$$

Proof. For any Δ,

$\Gamma \Rightarrow\ \sim A$	Hypothesis
$\Gamma \Rightarrow\ \sim B$	Hypothesis
$\Pr(\sim A, \Gamma \cup \Delta) = 1$	Definition of validity
$\Pr(\sim B, \Gamma \cup \Delta) = 1$	Definition of validity
$\Pr(A, \Gamma \cup \Delta) = 0$	Theorem 2.12
$\Pr(B, \Gamma \cup \Delta) = 0$	Theorem 2.12
$\Pr(A \vee B, \Gamma \cup \Delta) = 0$	POS.5, POS.6 and algebra
$\Pr(\sim(A \vee B), \Gamma \cup \Delta) = 1$	POS.9 (1)
$\Gamma \Rightarrow\ \sim(A \vee B)$	Definition of validity

$\square$

$$\frac{\Gamma \Rightarrow A \quad B, \Gamma \Rightarrow\ C}{A \supset B, \Gamma \Rightarrow\ C}\ L \supset$$

Proof. For any Δ,

$\Gamma \Rightarrow A$	Hypothesis
$B, \Gamma \Rightarrow\ C$	Hypothesis
$\Pr(A, \Gamma \cup \Delta) = 1$	Definition of validity
$\Pr(C, \Gamma \cup \Delta \cup \{B\}) = 1$	Definition of validity
$\Pr(C, \Gamma \cup \Delta \cup \{B\}) =$ $\Pr(C, \Gamma \cup \Delta \cup \{(A \supset B)\})$	POS.12
$\Pr(C, \Gamma \cup \Delta \cup \{(A \supset B)\}) = 1$	Algebra
$A \supset B, \Gamma \Rightarrow\ C$	Definition of validity

$\square$

$$\frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \supset B}\ R \supset$$

Proof.

$A, \Gamma \Rightarrow B$	Hypothesis
$\Pr(B, \Gamma \cup \Delta \cup \{A\}) = 1$	Definition of validity
$\Pr(A \supset B, \Gamma \cup \Delta) = 1$	POS.8
$\Gamma \Rightarrow A \supset B$	Definition of validity

□

$$\frac{A, \sim B, \Gamma \Rightarrow C}{\sim(A \supset B), \Gamma \Rightarrow C}\ L{\sim}{\supset}$$

Proof. For any Δ,

$A, \sim B, \Gamma \Rightarrow C$	Hypothesis
$\Pr(C, \Gamma \cup \Delta \cup \{A, \sim B\} = 1$	Definition of validity
$\Pr(C, \Gamma \cup \Delta \cup \{A \wedge \sim B\} = 1$	POS.10
$\Pr(C, \Gamma \cup \Delta \cup \{\sim(A \supset B)\} = 1$	Theorem 2.6
$\sim(A \supset B), \Gamma \Rightarrow C$	Definition of validity

□

$$\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow \sim B}{\Gamma \Rightarrow \sim(A \supset B)}\ R{\sim}{\supset}$$

Proof. For any Δ,

$\Gamma \Rightarrow A$	Hypothesis
$\Gamma \Rightarrow \sim B$	Hypothesis
$\Pr(A, \Gamma \cup \Delta) = 1$	Definition of validity
$\Pr(\sim B, \Gamma \cup \Delta) = 1$	Definition of validity
$\Pr(\sim(A \supset B), \Gamma \cup \Delta) =$	
$\quad \Pr(A, \Gamma \cup \Delta) \times \Pr(\sim B, \Gamma \cup \Delta \cup \{A\})$	POS.9 (2)
$\Pr(\sim B, \Gamma \cup \Delta) = \Pr(\sim B, \Gamma \cup \Delta \cup \{A\}) = 1$	Theorem 2.11
$\Pr(\sim(A \supset B), \Gamma \cup \Delta) = 1$	Algebra
$\Gamma \Rightarrow \sim(A \supset B)$	Definition of validity

□

$$\frac{A[t|x], \Gamma \Rightarrow C}{\forall x A, \Gamma \Rightarrow C}\ L\,\forall$$

Proof. We have to show that for any $\Pr, A, C, t, \Gamma$, if $\Pr(C, \Gamma \cup \{A[t|x]\} \cup \Delta) = 1$ for all Δ, then $\Pr(C, \Gamma \cup \{\forall x A\} \cup \Delta) = 1$ for all Δ.
This corresponds exactly to what POS.18 says.

□

$$\frac{\Gamma \Rightarrow A[y|x]}{\Gamma \Rightarrow \forall x A}\ R\,\forall$$ where y is not free in Γ and y is x or y is not free in A.

Proof. Let $t_1, \ldots, t_n, \ldots$ be an enumeration of all the terms that are free for x in A.

$\Gamma \Rightarrow A[y\vert x]$	Hypothesis
$\Pr(A[y\vert x], \Gamma \cup \Delta) = 1$	Definition of validity
$\Pr(A[t_1\vert x], \Gamma \cup \Delta) = 1$	POS.20
.	.
.	.
.	.
$\Pr(A[t_n\vert x], \Gamma \cup \Delta) = 1$	POS.20
$\Pr(A[t_1\vert x] \wedge \ldots \wedge, A[t_1\vert x], \Gamma \cup \Delta) = 1$	DF.10 n times, Thm 2.11
$\lim\limits_{n\to\infty} \Pr((A[t_{i_1}\vert x] \wedge \ldots \wedge A[t_{i_n}\vert x]), \Gamma \cup \Delta) = 1$	Elementary calculus
$\Pr(\forall x A, \Gamma \cup \Delta)) = 1$	POS.16
$\Gamma \Rightarrow \forall x A$	Definition of validity

□

$$\frac{A[y|x], \Gamma \Rightarrow C}{\exists x A, \Gamma \Rightarrow C}\ L\,\exists$$ where y is not free in Γ and C and y is x or y is not free in$A$.

Proof. This corresponds exactly to what POS.19 says.

□

$$\frac{\Gamma \Rightarrow A[t|x]}{\Gamma \Rightarrow \exists x A}\ R\,\exists$$

Proof. The proof is quite similar to that of $R\,\forall$ and left to the reader.

□

The proofs for the rules with strong negation of quantifiers are dual of the preceding ones and are left to the reader.

This completes the proof of soundness.

□

4 Completeness

The strategy we use to prove completeness is the following. Following Kripke's idea [6, 1, 5] for designing models for intuitionistic logic, we define a 3-valued model$\{1, 0, u\}$ where the three values are standing respectively for true, false and unknown. Counter to what we have said at the beginning of this paper, in this very particular model structure, u can be considered as a value. We then show that any consistent set of sentences of SCILSN admits a 3-valued model. Using this model, we define partial conditional probability functions taking only the three values and we finally show that these functions satisfy DF.1-DF.2 and POS.3-POS.20. Calling these partial probability functions $partial\, opinated\, functions$, we show that every consistent set defines a partial opinated function, which is stronger than to show that every consistent set defines a partial probability function. Moreover, this model is canonical : if A and Γ are such that $\Gamma \not\Rightarrow A$, then there is a Δ such that $\Pr(A, \Gamma \cup \Delta) \neq 1$.

Definition 4.1. *A Deductively Closed Saturated Set (DCSS) is a set of wffs Δ such that*
(i) If $\Delta \Rightarrow A$, then $A \in \Delta$ (closure);
(ii) If $A \vee B \in \Delta$, then $A \in \Delta$ or $B \in \Delta$ (saturation);
(iii) It is not the case that $\Delta \Rightarrow F$ (consistency).

Definition 4.2. *If Γ is consistent, $U(\Gamma) = \{\Delta : \Delta\, is\, a\, DCSS\, and\, \Delta \subseteq \Gamma\}$.*

Theorem 4.3. *If $\Gamma \not\Rightarrow A$, there is a DCSS Δ such that $\Gamma \subseteq \Delta$ and $\Delta \not\Rightarrow A$.*

Proof. (This proof is not constructive.)
A is called the $test\ formula$. Let $E =< E_0, E_1, E_2, \ldots >$ be an enumeration of all wffs where each wff appears denumerably many times. We define the following sequence of sets:

$\Gamma_0 = \Gamma$;

.

.

.

$\Gamma_{k+1} = \Gamma_k$ if $\Gamma_k \cup \{E_k\} \Rightarrow A$;

$\Gamma_{k+1} = \Gamma_k \cup \{E_k\}$ if $\Gamma_k \Rightarrow E_k$, and E_k is not $(B \vee C)$;

if E_k is $(B \vee C)$,

$\Gamma_{k+1} = \Gamma_k \cup \{E_k\} \cup \{B\}$ if $\Gamma_k \cup \{E_k\} \cup \{B\} \not\Rightarrow A$
else $\Gamma_{k+1} = \Gamma_k \cup \{E_k\} \cup \{C\}$.

We define

$$\Delta = \bigcup_{k=0}^{\infty} \Gamma_k$$

Claim

(1) $\Delta \not\Rightarrow A$

We first show that, for any k, $\Gamma_k \not\Rightarrow A$.

For $k = 0$, it is trivial. Let us suppose that $\Gamma_k \not\Rightarrow A$, we show that $\Gamma_{k+1} \not\Rightarrow A$.

If $\Gamma_{k+1} = \Gamma_k \cup \{E_k\}$ because $\Gamma_k \Rightarrow E_k$, and E_k is not $(B \vee C)$, we get the result by the Cut rule.

Let us suppose that E_k is $(B \vee C)$.

If $\Gamma_{k+1} = \Gamma_k \cup \{E_k\} \cup \{B\}$ because $\Gamma_{k+1} \not\Rightarrow A$, it is trivial;

If $\Gamma_{k+1} = \Gamma_k \cup \{E_k\} \cup \{C\}$ because $\Gamma_k \cup \{E_k\} \cup \{B\} \Rightarrow A$, we have to show that $\Gamma_k \cup \{E_k\} \cup \{C\} \not\Rightarrow A$.

Let us suppose that $\Gamma_k \cup \{E_k\} \cup \{C\} \Rightarrow A$.

From $L\vee$ we have:

$\Gamma_k \cup \{E_k\} \cup \{(B \vee C)\} \Rightarrow A$. But E_k is $(B \vee C)$, so $\Gamma_k \cup \{E_k\} \Rightarrow A$ which contradicts the hypothesis.

(2) If $\Delta \Rightarrow B$, then $B \in \Delta$ because B is one of the E_k.

(3) Δ is saturated, i.e., if $A \vee B \in \Delta$, then $A \in \Delta$ or $B \in \Delta$. It is a trivial consequence of the definition of Δ.

(4) Δ is consistent. This follows from the fact that $\Delta \not\Rightarrow A$.

$\square$

The most interesting consequence of Theorem 4.3 is that if A is a classical tautology and Γ is such that $\Gamma \not\Rightarrow A$, then there is a DCSS Δ such that $\Gamma \subseteq \Delta$ and $\Delta \not\Rightarrow A$.

Corollary 4.4. *If Γ is consistent, there is a DCSS Δ such that $\Gamma \subseteq \Delta$.*

Proof. As Γ is consistent, there is a A such that $\Gamma \not\Rightarrow A$. We define a *DCSS* Δ starting from Γ using A as the test formula. □

Corollary 4.5. *Let Γ be a consistent set. If there is no DCSS Δ such that $\Gamma \subseteq \Delta$ and $A \in \Delta$, then $\Gamma \cup \{A\}$ is inconsistent.*

Proof. This is a trivial consequence of Corollary 4.4. □

Corollary 4.6. *If $A \in \Delta$ for any DCSS Δ such that $\Gamma \subseteq \Delta$, then $\Gamma \Rightarrow A$.*

Proof. The above is the contraposition of Theorem 4.3. □

Theorem 4.7. *Let W be the set of all DCSS and $\Delta \in W$. If $A \supset B \in \Delta$ and $\Delta' \in W$ with $\Delta \subseteq \Delta'$ and $A \in \Delta'$, then $B \in \Delta'$.*

Proof. Let $\Delta, \Delta' \in W$, $A \supset B \in \Delta$, $\Delta \subseteq \Delta'$ and $A \in \Delta'$. We have $A \supset B \in \Delta'$ and by

$$\frac{\Delta' \Rightarrow A \qquad B, \Delta' \Rightarrow B}{A \supset B, \Delta' \Rightarrow B} L \supset$$

and by closure $B \in \Delta'$. □

Theorem 4.8. *Let Γ be a consistent set of wffs such that $\Gamma \not\Rightarrow A \supset B$. Then there is a DCSS Δ such that $\Gamma \subseteq \Delta$, $A \in \Delta$ and $B \notin \Delta$.*

Proof. We have $A, \Gamma \not\Rightarrow B$, Otherwise, by $R \supset$, $\dfrac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \supset B}$ which contradicts the hypothesis. We then start again the Theorem 4.3 using $\Gamma_0 = \Gamma \cup \{A\}$ and B as test formula. □

Definition 4.9. *Let Γ be a consistent set of wffs. $U(\Gamma) = \{\Delta : \Gamma \subseteq \Delta$ and Δ is a DCSS.*

Theorem 4.10. *Let $\Delta \in U(\Gamma)$. If $B \in \Delta$, then $\Delta \in U(\Gamma \cup \{B\})$.*

Proof. (This proof is not constructive)
If $B \in \Gamma$, the proof is trivial. Let us suppose it is not the case that $B \in \Gamma$. Let $E =< E_0, E_1, E_2, \ldots >$ be an enumeration of all the wffs of Δ where every $A \in \Delta$ appears denumerably many times. Let us consider the following two sequences:

$$
\begin{array}{ll}
\Delta_0 = \Gamma & \Delta'_0 = \Gamma \cup \{B\} \\
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \cdot \\
\Delta_{i+1} = \Delta_i \cup \{E_i\} & \Delta_{i+1} = \Delta'_i \cup \{E_i\} \\
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \cdot
\end{array}
$$

It is clear that $\Delta = \bigcup_{k=0}^{\infty} \Delta_k$.

It is also clear that, for any i, $\Delta_i \subseteq \Delta'_i$.
Let k be the smallest integer such that E_k is B. We then have $\Delta_{k+1} = \Delta'_{k+1}$ and for any $k' \geq k+1$, $\Delta_{k'} = \Delta'_{k'}$.

$$\Delta' = \bigcup_{k=0}^{\infty} \Delta'_k = \Delta. \text{ But } \Delta' \in U(\Gamma \cup \{B\}).$$

□

Corollary 4.11. *If $A \in \Delta$ for any $\Delta \in U(\Gamma)$, then $U(\Gamma) = U(\Gamma \cup \{A\})$.*

Proof. It is a trivial consequence of Theorem 4.10. □

Theorem 4.12. *Let Δ be a DCSS such that A, $A \supset B \in \Delta$. Then $B \in \Delta$.*

Proof. The above is a trivial consequence of Theorem 4.7. □

Theorem 4.13. *For any Γ and any $A_1, \ldots, A_n$, $(A_1 \wedge \ldots \wedge A_n), \Gamma \Rightarrow C$ iff $A_1, \ldots, A_n, \Gamma \Rightarrow C$*

Proof. We just give a sketch of the proof which is quite trivial.
$\rightarrow$

We have:

$$\frac{A_1,\ldots,A_n,\Gamma \Rightarrow A_1 \qquad A_1,\ldots,A_n,\Gamma \Rightarrow A_2}{A_1,\ldots,A_n,\Gamma \Rightarrow A_1 \wedge A_2,}\ R\wedge$$

$$\frac{A_1,\ldots,A_n,\Gamma \Rightarrow A_1 \wedge A_2 \qquad A_1,\ldots,A_n,\Gamma \Rightarrow A_3}{A_1,\ldots,A_n,\Gamma \Rightarrow A_1 \wedge A_2 \wedge A_3}\ R\wedge$$

After $(n-1)$ steps, we get:

$$\frac{A_1,\ldots,A_n,\Gamma \Rightarrow A_1 \wedge A_2 \wedge \ldots \wedge A_{n-1} \qquad A_1,\ldots,A_n,\Gamma \Rightarrow A_n}{A_1,\ldots,A_n,\Gamma \Rightarrow A_1 \wedge \ldots \wedge A_n}\ R\wedge$$

By the Cut rule, if $(A_1 \wedge \ldots \wedge A_n),\Gamma \Rightarrow C$, then $A_1,\ldots,A_n,\Gamma \Rightarrow C$.

$\leftarrow$

Let us suppose that $A_1,\ldots,A_n,\Gamma \Rightarrow C$. We have

$$\frac{A_1,\ldots,A_{n-2},A_{n-1},A_n,\Gamma \Rightarrow C}{A_1,\ldots,A_{n-2},(A_{n-1} \wedge A_n),\Gamma \Rightarrow C}\ L\wedge$$

Applying $L\wedge$ $(n-1)$ times, we get

$$\frac{A_1,\ldots,A_n,\Gamma \Rightarrow C}{(A_1 \wedge \ldots \wedge A_n),\Gamma \Rightarrow C}\ L\wedge$$

$\square$

Theorem 4.14. *For any* Γ *and any* $A_1,\ldots,A_n$, $(A_1 \vee \ldots \vee A_n),\Gamma \Rightarrow C$ *iff for any* i, $1 \leq i \leq n$, $A_i,\Gamma \Rightarrow C$

The proof is quite elementary and is left to the reader.

Theorem 4.15. $\dfrac{(\sim A_1 \wedge \ldots \wedge \sim A_n),\Gamma \Rightarrow C}{(\sim(A_1 \vee \ldots \vee A_n)),\Gamma \Rightarrow C}$ *and* $\dfrac{(\sim(A_1 \vee \ldots \vee A_n)),\Gamma \Rightarrow C}{(\sim A_1 \wedge \ldots \wedge \sim A_n),\Gamma \Rightarrow C}$

We merely give a sketch of the proof. By Theorem 4.13 we have, if $A_1,\ldots,A_n,\Gamma \Rightarrow C$ then $(A_1 \wedge \ldots \wedge A_n),\Gamma \Rightarrow C$.
We show that $\sim A_1,\ldots,\sim A_n,\Gamma \Rightarrow (\sim(A_1 \vee \ldots \vee A_n))$ and the result follows by

the Cut rule.
By $R{\sim}\vee$, we have ${\sim}A_1, \ldots, {\sim}A_n, \Gamma \Rightarrow ({\sim}(A_1 \vee A_2))$. Applying $R{\sim}\vee$ $(n-1)$ times, we get the result we are looking for.

For the converse, we have to show that $({\sim}(A_1 \vee \ldots \vee A_n)), \Gamma \Rightarrow ({\sim}A_1 \wedge \ldots \wedge {\sim}A_n)$. In order to do this, we proceed in two steps. We first show that if ${\sim}A_1, \ldots, {\sim}A_n, \Gamma \Rightarrow C$ then $({\sim}(A_1 \vee \ldots \vee A_n)), \Gamma \Rightarrow C$ using $L{\sim}\vee$ $(n-1)$ times. Then we use Theorem 4.13 which implies that ${\sim}A_1, \ldots, {\sim}A_n, \Gamma \Rightarrow ({\sim}A_1 \wedge \ldots \wedge {\sim}A_n)$.

□

Theorem 4.16. $\dfrac{({\sim}A_1 \vee \ldots \vee {\sim}A_n), \Gamma \Rightarrow C}{({\sim}(A_1 \wedge \ldots \wedge A_n)), \Gamma \Rightarrow C}$ *and* $\dfrac{({\sim}(A_1 \wedge \ldots \wedge A_n)), \Gamma \Rightarrow C}{({\sim}A_1 \vee \ldots \vee {\sim}A_n), \Gamma \Rightarrow C}$

The proof is left to the reader.

Theorem 4.17. *If $U(\Gamma) = U(\Gamma')$, then $U(\Gamma \cup \{A\}) = U(\Gamma' \cup \{A\})$.*

Proof. Let $\Delta \in U(\Gamma$ and $E =< E_0, E_1, E_2, \ldots >$ be an enumeration of all the wffs where every wff appears denumerably many times.
We define the following sequence $\Lambda_0,\ \Lambda_1,\ \Lambda_2,\ \ldots\ \Lambda_n,\ \ldots$:

$$
\begin{aligned}
&\Lambda_0 = \Delta \cup \{A\};\\
&\quad .\\
&\quad .\\
&\quad .\\
&\Lambda_{n+1} = \Lambda_n \cup \{E_n\} \text{ if } \Lambda_n \Rightarrow E_n \text{ and } E_n \text{ is not } (B \vee C);\\
&\Lambda_{n+1} = \Lambda_n \cup \{E_n\} \cup \{B\} \text{ if } \Lambda_n \Rightarrow E_n \text{ and } E_n \text{ is } (B \vee C) \text{ and}\\
&\qquad\quad \Lambda_n \cup \{E_n\} \cup \{B\} \text{ is consistent};\\
&\Lambda_{n+1} = \Lambda_n \cup \{E_n\} \cup \{C\} \text{ otherwise};\\
&\quad .\\
&\quad .\\
&\quad .
\end{aligned}
$$

Let $\Lambda_\Delta = \bigcup\limits_{n=0}^{\infty} \Lambda_n$

Claim: $\Lambda \in U(\Gamma \cup \{A\})$.

It is clear that Λ_Δ is a DCSS and that $\Gamma \cup \{A\} \in \Lambda_\Delta$. As $U(\Gamma) = U(\Gamma')$, a similar argument leads us to conclude that $\Gamma' \cup \{A\} \in \Lambda_\Delta$.

In order to conclude that $U(\Gamma \cup \{A\}) = U(\Gamma' \cup \{A\})$, we have to show that, for any $\Lambda \in U(\Gamma \cup \{A\})$, there is a $\Delta' \in U(\Gamma)$ such that $\Lambda = \Lambda_{\Delta'}$.
Let $\Lambda \in U(\Gamma \cup \{A\})$ and $E' =< E'_0, E'_1, E'_2, \ldots >$ be an enumeration of the wffs of Λ where every wff appears denumerably many times. We define the following sequence of sets:

$$\Lambda'_0 = \emptyset \text{ if } E'_0 \Rightarrow A,\ \Lambda'_0 = \{E'_0\} \text{ otherwise;}$$

$$\vdots$$

$$\Lambda'_{n+1} = \Lambda'_n \text{ if } \Lambda'_n \cup \{E'_n\} \Rightarrow A,\ \Lambda'_n \cup \{E'_n\} \text{ otherwise;}$$

$$\vdots$$

Let $\Lambda' = \bigcup_{n=0}^{\infty} \Lambda'_n$

Claims:
(1) $\Lambda' \in U(\Gamma)$
(2) If $\Lambda' \Rightarrow B$ then $B \in \Lambda'$
(3) If $(B \vee C) \in \Lambda'$, then $B \in \Lambda'$ or $C \in \Lambda'$

(1) $\Lambda' \in U(\Gamma)$	Trivial
(2) If $\Lambda' \Rightarrow B$ then $B \in \Lambda'$	
$\Lambda' \Rightarrow B$	Assumption
There is a Λ'_n such that $\Lambda'_n \Rightarrow B$	A proof in Λ' is finite.
There is a m such that B is E'_{n+m}	Definition of E'
$B \in \Lambda'_{n+m+1}$	Definition of Λ'_{n+m+1}
$B \in \Lambda'$	Definition of Λ'
(3) $(B \vee C) \in \Lambda'$	Assumption
$B \notin \Lambda'$ and $C \notin \Lambda'$	Assumption
$\Lambda' \cup \{B\} \Rightarrow A$ and $\Lambda' \cup \{C\} \Rightarrow A$	Definition of Λ'
$\Lambda' \cup \{(B \vee C)\} \Rightarrow A$	$L \vee$
$A \in \Lambda'$	Contradiction

□

Definition 4.18. *The pair $< W, \subseteq >$ is called the Kripkean canonical frame.*

Definition 4.19. *Let $A*$ be B or $\sim B$ according to whether A is $\sim B$ or B.*

Definition 4.20. *The canonical partial probabilistic model is the 3-uple $< W, \subseteq, \mathrm{Pr}_{<W,\subseteq>} >$ where $\mathrm{Pr}_{<W,\subseteq>} : L \times 2^L \to \{0, 1, u\}$ is such that, for any A, Γ*

(i) If A is a literal,

$$\mathrm{Pr}_{<W,\subseteq>}(A,\Gamma) = \begin{cases} 1 \text{ if } A \in \Delta \text{ for any } \Delta \in U(\Gamma) \\ 0 \text{ if } A* \in \Delta \text{ for any } \Delta \in U(\Gamma) \\ u \text{ otherwise} \end{cases}$$

(ii) If $\sim\sim B$,

$$\mathrm{Pr}_{<W,\subseteq>}(A,\Gamma) = \begin{cases} 1 \text{ if } B \in \Delta \text{ for any } \Delta \in U(\Gamma) \\ 0 \text{ if } \sim B \in \Delta \text{ for any } \Delta \in U(\Gamma) \\ u \text{ otherwise} \end{cases}$$

(iii) If A is $B \wedge C$,

$$\mathrm{Pr}_{<W,\subseteq>}(A,\Gamma) = \begin{cases} 1 \text{ if for any } \Delta \in U(\Gamma),\ B \in \Delta \text{ and } C \in \Delta \\ 0 \text{ if for any } \Delta \in U(\Gamma),\ \sim B \in \Delta \text{ or } \sim C \in \Delta \\ u \text{ otherwise} \end{cases}$$

(iv) If A is $\sim(B \wedge C)$,

$$\mathrm{Pr}_{<W,\subseteq>}(A,\Gamma) = \begin{cases} 1 \text{ if for any } \Delta \in \mathrm{U}(\Gamma), \sim B \in \Delta \text{ or } \sim C \in \Delta \\ 0 \text{ if for any } \Delta \in \mathrm{U}(\Gamma), B \in \Delta \text{ and } C \in \Delta \\ u \text{ otherwise} \end{cases}$$

(v) If A is $B \vee C$,

$$\mathrm{Pr}_{<W,\subseteq>}(A,\Gamma) = \begin{cases} 1 \text{ if for any } \Delta \in U(\Gamma),\ B \in \Delta \text{ or } C \in \Delta \\ 0 \text{ if for any } \Delta \in U(\Gamma),\ \sim B \in \Delta \text{ and } \sim C \in \Delta \\ u \text{ otherwise} \end{cases}$$

(vi) If A is $\sim(B \vee C)$,

$$\Pr_{<W,\subseteq>}(A,\Gamma) = \begin{cases} 1 \text{ if for any } \Delta \in U(\Gamma),\ \sim B \in \Delta \text{ and} \sim C \in \Delta \\ 0 \text{ if for any } \Delta \in U(\Gamma),\ B \in \Delta \text{ or } C \in \Delta \\ u \text{ otherwise} \end{cases}$$

(vii) If A is $B \supset C$,

$$\Pr_{<W,\subseteq>}(A,\Gamma) = \begin{cases} 1 \text{ if for any } \Delta \in U(\Gamma) \text{ such that } B \in \Delta, C \in \Delta \\ 0 \text{ if for any } \Delta \in U(\Gamma), B \in \Delta \text{ and} \sim C \in \Delta \\ u \text{ otherwise} \end{cases}$$

(viii) If A is $\sim(B \supset C)$,

$$\Pr_{<W,\subseteq>}(A,\Gamma) = \begin{cases} 1 \text{ if for any } \Delta \in U(\Gamma),\ B \in \Delta \text{ and} \sim C \in \Delta \\ 0 \text{ if for any } \Delta \in U(\Gamma),\ \sim B \in \Delta \text{ or } C \in \Delta \\ u \text{ otherwise} \end{cases}$$

(ix) If A is $\forall xB$,

$$\Pr_{<W,\subseteq>}(A,\Gamma) = \begin{cases} 1 \text{ if for any } \Delta \in U(\Gamma),\ B[y|x] \in \Delta \text{ for any } y \text{ not free in } \forall xB \\ 0 \text{ if for any } \Delta \in U(\Gamma),\ \sim B[t|x] \in \Delta \text{ for some } t \text{ free for } x \text{ in } B \\ u \text{ otherwise} \end{cases}$$

(x) If A is $\sim\forall xB$,

$$\Pr_{<W,\subseteq>}(A,\Gamma) = \begin{cases} 1 \text{ if for any } \Delta \in U(\Gamma),\ \sim B[t|x] \in \Delta \text{ for some } t \text{ free for } x \text{ in } B \\ 0 \text{ if for any } \Delta \in U(\Gamma),\ B[y|x] \in \Delta \text{ for any } t \text{ free in } \forall xB \\ u \text{ otherwise} \end{cases}$$

(xi) If A is $\exists xB$,

$$\Pr_{<W,\subseteq>}(A,\Gamma) = \begin{cases} 1 \text{ if for any } \Delta \in U(\Gamma),\ B[t|x] \in \Delta \text{ for some } t \text{ free for } x \text{ in } B \\ \\ 0 \text{ if for any } \Delta \in U(\Gamma),\ {\sim}B[y|x] \in \Delta \text{ for any } y \text{ not free in } \exists xB \\ \\ u \text{ otherwise} \end{cases}$$

(xii) If A is ${\sim}\exists xB$,

$$\Pr_{<W,\subseteq>}(A,\Gamma) = \begin{cases} 1 \text{ if for any } \Delta \in U(\Gamma),\ {\sim}B[y|x] \in \Delta \text{ for any } y \text{ free } x \text{ in } {\sim}\exists xB \\ \\ 0 \text{ if for any } \Delta \in U(\Gamma),\ B[t|x] \in \Delta \text{ for some } t \text{ free for } x \text{ in } B \\ \\ u \text{ otherwise} \end{cases}$$

Theorem 4.21. *(We drop the index.) For any* Pr*, A and Γ, if, for any $\Delta \in U(\Gamma), A \in \Delta$, then* $\Pr(A,\Gamma) = 1$

Proof.
(i) A is a literal. It is trivial by definition 4.20(i).

(ii) A is ${\sim}{\sim}B$. By definition 4.20(ii), $\Pr(A,\Gamma) = 1$ if $B \in \Delta$ for any $\Delta \in U(\Gamma)$. But by $R{\sim}{\sim}$, ${\sim}{\sim}B \in \Delta$.

(iii) A is $B \wedge C$. By definition 4.20(iii), $\Pr(A,\Gamma) = 1$ if $B \in \Delta$ and $C \in \Delta$ for any $\Delta \in U(\Gamma)$. In that case, by $R\wedge$, $B \wedge C \in \Delta$ for any $\Delta \in U(\Gamma)$.

(iv) A is ${\sim}(B \wedge C)$. By definition 4.20(iv), $\Pr(A,\Gamma) = 1$ if ${\sim}B \in \Delta$ or ${\sim}C \in \Delta$ for any $\Delta \in U(\Gamma)$. In that case, by $R{\sim}\wedge_1$ or $R{\sim}\wedge_2$, ${\sim}(B \wedge C) \in \Delta$ for any $\Delta \in U(\Gamma)$.

(v) A is $B \vee C$. By definition 4.20(v), $\Pr(A,\Gamma) = 1$ if $B \in \Delta$ or $C \in \Delta$ for any $\Delta \in U(\Gamma)$. In that case, by $R\vee_1$ or $R\vee_2$, $B \vee C \in \Delta$ for any $\Delta \in U(\Gamma)$.

(vi) A is ${\sim}(B \vee C)$. By definition 4.20(vi), $\Pr(A,\Gamma) = 1$ if ${\sim}B \in \Delta$ and ${\sim}C \in \Delta$ for any $\Delta \in U(\Gamma)$. In that case, by $R{\sim}\vee$, ${\sim}(B \vee C) \in \Delta$ for any $\Delta \in U(\Gamma)$.

(vii) A is $B \supset C$. By definition 4.20(vii), $\Pr(A,\Gamma) = 1$ if for any $\Delta \in U(\Gamma)$

such that if $B \in \Delta$, then $C \in \Delta$. In that case, by $R \supset$, for any Δ such that $B \in \Delta$, $B \supset C \in \Delta$.

(viii) A is $\sim(B \supset C)$. By definition 4.20(viii), $\Pr(A, \Gamma) = 1$ if $B \in \Delta$ and $\sim C \in \Delta$ for any $\Delta \in U(\Gamma)$. In that case, by $R{\sim}\supset$, $\sim(B \supset C) \in \Delta$ for any $\Delta \in U(\Gamma)$. □

Theorem 4.22. *For any* Pr*, A and Γ, if, for any $\Delta \in U(\Gamma), \sim A \in \Delta$, then* $\Pr(A, \Gamma) = 0$.

Proof.
(i) A is a literal. If A is p, then A^* is $\sim p \in \Delta$. If A is $\sim p$, A^* is p and by $R{\sim}{\sim}$, $\sim\sim p \in \Delta$ i.e. $\sim A \in \Delta$.

(ii) A is $\sim\sim B$. By definition 4.20(ii), $\Pr(A, \Gamma) = 0$ if $\sim B \in \Delta$ for any $\Delta \in U(\Gamma)$. By $R{\sim}{\sim}$, $\sim\sim\sim B \in \Delta$ for any $\Delta \in U(\Gamma)$ i.e. $\sim A \in \Delta$.

(iii) A is $B \wedge C$. By definition 4.20(iii), $\Pr(A, \Gamma) = 0$ if $\sim B \in \Delta$ or $\sim C \in \Delta$ for any $\Delta \in U(\Gamma)$. By $R{\sim}\wedge_1$ or $R{\sim}\wedge_2$, $\sim(B \wedge C) \in \Delta$ for any $\Delta \in U(\Gamma)$.

(iv) A is $\sim(B \wedge C)$. By definition 4.20(iv), $\Pr(A, \Gamma) = 0$ if $B \in \Delta$ and $C \in \Delta$ for any $\Delta \in U(\Gamma)$. In that case, by $R\wedge$, $B \wedge C \in \Delta$ for any $\Delta \in U(\Gamma)$ and by $R{\sim}{\sim}$, $\sim\sim(B \wedge C) \in \Delta$ for any $\Delta \in U(\Gamma)$.

(v) A is $B \vee C$. By definition 4.20(v), $\Pr(A, \Gamma) = 0$ if $\sim B \in \Delta$ and $\sim C \in \Delta$ for any $\Delta \in U(\Gamma)$. In that case, by $R{\sim}\vee$, $\sim(B \vee C)$.

(vi) A is $\sim(B \vee C)$. By definition 4.20(vi), $\Pr(A, \Gamma) = 0$ if $B \in \Delta$ or $C \in \Delta$ for any $\Delta \in U(\Gamma)$. In that case, by $R\vee_1$ or $R\vee_2$, $B \vee C \in \Delta$ for any $\Delta \in U(\Gamma)$ and by $R{\sim}{\sim}$, $\sim\sim(B \vee C) \in \Delta$ for any $\Delta \in U(\Gamma)$.

(vii) A is $B \supset C$. By definition 4.20(vii), $\Pr(A, \Gamma) = 0$ if $B \in \Delta$ and $\sim C \in \Delta$ for any $\Delta \in U(\Gamma)$. In that case, by $R\wedge$, $B \wedge C \in \Delta$ for any $\Delta \in U(\Gamma)$. In that case, by $R{\sim}\supset$, $\sim(B \supset C) \in \Delta$ for any $\Delta \in U(\Gamma)$.

□

Theorem 4.23. *For any* Pr*,A and Γ,* $\Pr(A, \Gamma) = 1$ *iff for any $\Delta \in U(\Gamma), A \in \Delta$ and* $\Pr(A, \Gamma) = 0$ *iff or any $\Delta \in U(\Gamma), \sim A \in \Delta$.*

Proof. The ifs come from Theorem 4.21 and Theorem 4.22. The only ifs come from the u otherwise clause of definition 4.20. □

Theorem 4.24. *In the canonical model, for any* Γ *and any* Pr, Γ *is consistent iff* Γ *is Pr-normal.*

Proof.
(1) If Γ is consistent, then $\Gamma \not\Rightarrow F$. Furthermore, $F \notin \Delta$ for any $\Delta \in U(\Gamma)$ and by Theorem 4.23, $\Gamma \Rightarrow \sim F$ and thus $\Pr(F, \Gamma) = 0$. So, Γ is Pr-normal.

(2) If Γ is inconsistent then $\Gamma \Rightarrow F$. By axiom 2, for any C, $F, \Gamma \Rightarrow C$ and by the Cut rule, $\Gamma \Rightarrow C$. By Theorem 4.21, $\Pr(C, \Gamma) = 1$ and Γ is Pr-abnormal. □

So, with respect to the canonical model, the two expressions are equivalent.

Theorem 4.25. *The canonical model gives to the connectives* $\wedge$ *and* $\vee$ *the value of the Kleene strong connectives in the following sense:*

$$\Pr_{<W,\subseteq>}(A \wedge B, \Gamma) = \begin{cases} 1 \text{ iff } \Pr_{<W,\subseteq>}(A, \Gamma) = \Pr_{<W,\subseteq>}(B, \Gamma) = 1 \\ 0 \text{ iff } \Pr_{<W,\subseteq>}(A, \Gamma) = 0 \text{ or } \Pr_{<W,\subseteq>}(B, \Gamma) = 0 \\ u \text{ otherwise} \end{cases}$$

and

$$\Pr_{<W,\subseteq>}(A \vee B, \Gamma) = \begin{cases} 1 \text{ iff } \Pr_{<W,\subseteq>}(A, \Gamma) = 1 \text{ or } \Pr_{<W,\subseteq>}(B, \Gamma) = 1 \\ 0 \text{ iff } \Pr_{<W,\subseteq>}(A, \Gamma) = \Pr_{<W,\subseteq>}(, \Gamma) = 0 \\ u \text{ otherwise} \end{cases}$$

Proof. The proof is straightforward using Theorem 4.21 and Theorem 4.22 and is left to the reader. □

Theorem 4.26. *If* $\Pr(A, \Gamma) = u$ *and* $\Pr(A, \Gamma \cup \{B\}) = 0$, *then* $\Gamma \cup \{A, B\})$ *is inconsistent.*

Proof. Let us suppose that $\Gamma \cup \{A, B\})$ is consistent. In that case $U(\Gamma \cup \{A, B\})$ is not empty. So there is a $\Delta \in U(\Gamma)$ which contains A and B and thus $\Pr(A, \Gamma \cup \{B\}) \neq 0$. □

We need to make sure of one last thing: Does $\Pr_{<W,\subseteq>}$ define a partial conditional probability function?

Theorem 4.27. $\Pr_{<W,\subseteq>}$ *satisfies DF.1-DF.2 and POS.3-POS.20.*

Proof. (We drop the index.)

Let us begin with

POS. 7 $\Pr(\bigwedge_{i=1}^{n} A_i, \Gamma) = \Pr(\bigwedge_{i=1}^{n} A_{per_n(i)}, \Gamma)$

We proceed by induction on the number of $\wedge$.

$n = 1$

We have to prove that, when both are defined, $\Pr(A_1 \wedge A_2, \Gamma) = \Pr(A_2 \wedge A_1, \Gamma)$.
We have two cases:
(1) $\Pr(A_1 \wedge A_2, \Gamma) = 1$ and
(2) $\Pr(A_1 \wedge A_2, \Gamma) = 0$

(1)

$\Pr(A_1 \wedge A_2, \Gamma) = 1$	Assumption
$(A_1 \wedge A_2) \in \Delta$ for any $\Delta \in U(\Gamma)$	Theorem 4.23
$A_1, A_2 \in \Delta$ for any $\Delta \in U(\Gamma)$	Definition 4.20(iii)
$(A_2 \wedge A_1) \in \Delta$ for any $\Delta \in U(\Gamma)$	$R\wedge$ and closure
$\Pr(A_2 \wedge A_1, \Gamma) = 1$	Theorem 4.23

(2)

$\Pr(A_1 \wedge A_2, \Gamma) = 0$	Assumption
$(\sim(A_1 \wedge A_2)) \in \Delta$ for any $\Delta \in U(\Gamma)$	Theorem 4.23
$A_1 \in \Delta or A_2 \in \Delta$ for any $\Delta \in U(\Gamma)$	Definition 4.20(iv)
$(A_2 \wedge A_1) \in \Delta$ for any $\Delta \in U(\Gamma)$	$R{\sim}\wedge_1$ or $R{\sim}\wedge_2$ and closure
$\Pr(A_2 \wedge A_1, \Gamma) = 0$	Theorem 4.23

Let us suppose it is the case for $n-1$ conjuncts. We have

$\Pr(\bigwedge_{i=1}^{n} A_i, \Gamma) = \Pr(A_1 \wedge (\bigwedge_{i=2}^{n} A_i, \Gamma)$	Definition of $\Pr(\bigwedge_{i=1}^{n} A_i, \Gamma)$
$= \Pr(A_{per_n(1)} \wedge (\bigwedge_{i=2}^{n} A_{per_n(i)}, \Gamma)$	Induction hypothesis
$= \Pr(\bigwedge_{i=1}^{n} A_{per_n(i)}, \Gamma)$	Algebra

DF. 1 If $\Pr(A_j, \Gamma) = 0$ for some $1 \leq j \leq n$, then $\Pr(\bigwedge_{i=1}^{n} A_i, \Gamma) = 0$;

Let per_n be a permutation such that $per_n(1) = j$

$\Pr(A_j, \Gamma) = 0$	Assumption
$\sim A_j \in \Delta$ for any $\Delta \in U(\Gamma)$	Theorem 4.23
$\sim(A_j \wedge (\bigwedge_{i=2}^{n} A_i, \Gamma)) \in \Delta$ for any $\Delta \in U(\Gamma)$	$R{\sim}_1$ + closure
$\sim(A_{per_n(1)} \wedge (\bigwedge_{i=2}^{n} A_{per_n(i)})) \in \Delta$ for any $\Delta \in U(\Gamma)$	$A_{per_n(1)} = A_j$
$\Pr(\bigwedge_{i=1}^{n} A_{per_n(i)}) = 0$	Theorem 4.23
$\Pr(\bigwedge_{i=1}^{n} A_i) = 0$	Pr satisfies DF. 7

DF. 2 If $\Pr(A_j, \Gamma) = 1$ for some $1 \leq j \leq n$, then $\Pr(\bigvee_{i=1}^{n} A_i, \Gamma) = 1$.

The proof is quite similar to the preceding one.

POS.3 $0 \leq \Pr(A, \Gamma) \leq 1$.
Trivial.

POS.4 If $A \in \Gamma$, then $\Pr(A, \Gamma) = 1$.
Trivial.

POS.6 $\Pr(\bigwedge_{i=1}^{n} A_i, \Gamma) = \Pr(A_1, \Gamma) \times \Pr(\bigwedge_{i=2}^{n} A_i, \Gamma \cup \{A_1\})$.

We have two cases.

(1) At least one of the A_j is such that $\Pr(A_j, \Gamma) = 0$. By the adequation of DF. 1, $\Pr(\bigwedge_{i=1}^{n} A_i, \Gamma) = 0$.

In that case, either $j = 1$ or $j \neq 1$. In both cases, $\Pr(A_1, \Gamma) \times \Pr(\bigwedge_{i=2}^{n} A_i, \Gamma \cup \{A_1\}) = 0$ because either $\Pr(A_1, \Gamma) = 0$ or, by the adequation of DF. 1 again, $\Pr(\bigwedge_{i=2}^{n} A_i, \Gamma \cup \{A_1\}) = 0$.

(2) All of the A_j's are such that $\Pr(A_j, \Gamma) = 1$.

In that subcase, by the definition of 4.20 (iii) for all j, $A_j \in \Delta$ for all $\Delta \in U(\Gamma)$ and applying $R\wedge$ $n-1$ times and by the closure, $\Pr(\bigwedge_{i=1}^{n} A_i, \Gamma) = 1$,

$\Pr(A_j, \Gamma) = 1$ and $\Pr(\bigwedge_{i=2}^{n} A_i, \Gamma \cup \{A_1\}) = \Pr(\bigwedge_{i=2}^{n} A_i, \Gamma) = 1$ (because $\Gamma = \Gamma \cup \{A_1\}$). We get $1 = 1 \times 1$ and we are done.

POS.5 $\Pr(\bigvee_{i=1}^{n} A_i, \Gamma) = \Pr(A_1, \Gamma) + \Pr(\bigvee_{i=2}^{n} A_i, \Gamma) - \Pr(A_1 \wedge (\bigvee_{i=2}^{n} A_i, \Gamma))$.

We merely have to verify all the possibilities when all the probabilities are defined. When $\Pr(A_1, \Gamma)$ and $\Pr(\bigvee_{i=2}^{n} A_i, \Gamma)$ are defined, $\Pr(A_1 \wedge (\bigvee_{i=2}^{n} A_i, \Gamma))$ and $Pr(\bigvee_{i=1}^{n} A_i, \Gamma)$ are also defined and by definition 4.20 (iii) and (v), we have the following table:

$\Pr(A_1, \Gamma)$	$\Pr(\bigvee_{i=2}^{n} A_i, \Gamma)$	$\Pr(A_1 \wedge (\bigvee_{i=2}^{n} A_i, \Gamma))$	$\Pr(\bigvee_{i=1}^{n} A_i, \Gamma)$
1	1	1	1
1	0	0	1
0	1	0	1
0	0	0	0

One can easily see that $\Pr(\bigvee_{i=1}^{n} A_i, \Gamma) = \Pr(A_1, \Gamma) + \Pr(\bigvee_{i=2}^{n} A_i, \Gamma) - \Pr(A_1 \wedge (\bigvee_{i=2}^{n} A_i, \Gamma))$.

POS.9 If Γ is Pr-normal, then $\Pr({\sim}A, \Gamma) =$
(1) $1 - \Pr(A, \Gamma)$ if A is an atom or F or $(B \wedge C)$ or $(B \vee C)$ or $\forall xB$ or $\exists xB$;
(2) $\Pr(B, \Gamma) \times \Pr({\sim}C, \Gamma \cup \{B\})$ if A is $(B \supset C)$;
(3) $\Pr(B, \Gamma)$ if A is ${\sim}B$.

(1) If A is an atom or F or $(B \wedge C)$ or $(B \vee C)$ or $\forall xB$ or $\exists xB$.

(i) $\Pr({\sim}p, \Gamma) = 1$

$\Pr({\sim}p, \Gamma) = 1$	Assumption
iff ${\sim}p \in \Delta$ for any $\Delta \in U(\Gamma)$	Theorem 4.23
iff $\Pr(p, \Gamma) = 0$	Theorem 4.20
iff $\Pr({\sim}p, \Gamma) = 1 - \Pr(p, \Gamma)$	Algebra

(ii) $\Pr(\sim p, \Gamma) = 0$

$\Pr(\sim p, \Gamma) = 0$	Assumption
iff $\sim\sim p \in \Delta$ for any $\Delta \in U(\Gamma)$	Theorem 4.23
iff $p \in \Delta$ for any $\Delta \in U(\Gamma)$	$L{\sim\sim}$ and $R{\sim\sim}$
iff $\Pr(p, \Gamma) = 1$	Theorem 4.20
iff $\Pr(\sim p, \Gamma) = 1 - \Pr(p, \Gamma)$	Algebra

(iii) If A is F
$\Pr(\sim F, \Gamma) = 1$ iff $\sim F \in \Delta$ for any $\Delta \in U(\Gamma)$ which is the case by **A3.**

But $\Pr(F, \Gamma) = 0$ iff $\sim F \in \Delta$ for any $\Delta \in U(\Gamma)$ which is the case by **A3.**
$\Pr(\sim F, \Gamma) = 1 - \Pr(F, \Gamma) = 1$ by algebra.

(iv)$\Pr(\sim F, \Gamma) = 1$ is not the case if Γ is consistent.

(v) We show that

(α) $\Pr(\sim(B \wedge C), \Gamma) = 1 - \Pr((B \wedge C), \Gamma)$

$\Pr(\sim(B \wedge C), \Gamma) = 1$	Assumption
$\sim(B \wedge C) \in \Delta$ for any $\Delta \in U(\Gamma)$	Theorem 4.23
$\Pr((B \wedge C), \Gamma) = 0$	Theorem 4.23
$1 = 1 - 0$	Algebra

$\Pr(\sim(B \wedge C), \Gamma) = 0$	Assumption
$(B \wedge C) \in \Delta$ for any $\Delta \in U(\Gamma)$	Theorem 4.23
$\Pr((B \wedge C), \Gamma) = 1$	Theorem 4.23
$0 = 1 - 1$	Algebra

(β) $\Pr(\sim(B \wedge C), \Gamma) = 1 - \Pr((B \wedge C), \Gamma)$

This case is as trivial as (α) and is left to the reader.

Cases (2) and (3) are also trivial.

POS.10 $\Pr(A, \Gamma \cup \{\bigwedge_{i=1}^{n} A_i\}) = \Pr(A, \Gamma \cup \{A_1, \ldots, A_n\})$

It is a straightforward consequence of Theorem 4.13.

POS.8 $\Pr(A \supset B, \Gamma) = \Pr(B, \Gamma \cup \{A\})$

We have to show that

(1) $\Pr(A \supset B, \Gamma) = 1$ iff $\Pr(B, \Gamma \cup \{A\}) = 1$

and

(2) $\Pr(A \supset B, \Gamma) = 0$ iff $\Pr(B, \Gamma \cup \{A\}) = 0$.

(1) We have to prove that, if $A \supset B \in \Delta$ for any $\Delta \in U(\Gamma)$ then $B \in \Delta$ for any $\Delta \in U(\Gamma \cup \{A\})$

$\Pr(A \supset B, \Gamma) = 1$	Assumption
$A \supset B \in \Delta$ for any $\Delta \in U(\Gamma)$	Theorem 4.23
$A \supset B, A \in \Delta$ for any $\Delta \in U(\Gamma \cup \{A\})$	$U(\Gamma \cup \{A\}) \subseteq U(\Gamma)$, $A \in \Gamma \cup \{A\}$
$B \in \Delta$ for any $\Delta \in U(\Gamma \cup \{A\})$	Theorem 4.12
$\Pr(B, \Gamma \cup \{A\}) = 1$	Theorem 4.23

We also have to prove the converse, i.e., if $\Pr(B, \Gamma \cup \{A\}) = 1$, then $\Pr(A \supset B, \Gamma) = 1$.

$\Pr(B, \Gamma \cup \{A\}) = 1$	Assumption
$B \in \Delta$ for any $\Delta \in U(\Gamma \cup \{A\})$	Theorem 4.23
$\Delta \cup \{A\} \Rightarrow B$ for any $\Delta \in U(\Gamma)$	Corollary 4.6
$\Delta \Rightarrow (A \supset B)$ for any $\Delta \in U(\Gamma)$	$R \supset$
$(A \supset B) \in \Delta$ for any $\Delta \in U(\Gamma)$	Δ is a $DCSS$
$\Pr(A \supset B, \Gamma) = 1$	Theorem 4.23

(2) We have to prove that
If $\Pr(A \supset B, \Gamma) = 0$, then $\Pr(B, \Gamma \cup \{A\}) = 0$.

$\Pr(A \supset B, \Gamma) = 0$	Assumption
$\sim(A \supset B) \in \Delta$ for any $\Delta \in U(\Gamma)$	Theorem 4.23
$A, \sim B \in \Delta$ for any $\Delta \in U(\Gamma \cup \{A\})$	PR.5 (2)
$\Pr(A, \Gamma) = 1$ and $\Pr(B, \Gamma) = 0$	Theorem 4.23
$\Pr(B, \Gamma \cup \{A\}) = 0$	Corollary 4.11

We also have to prove the converse.

$\mathrm{Pr}(B, \Gamma \cup \{A\}) = 0$	Assumption
$\sim B \in \Delta$ for any $\Delta \in U(\Gamma \cup \{A\})$	Theorem 4.23
$A \in \Delta$ for any $\Delta \in U(\Gamma \cup \{A\})$	$\Gamma \cup \{A\} \subseteq \Delta$
$A \wedge \sim B \in \Delta$ for any $\Delta \in U(\Gamma \cup \{A\})$	$R\wedge$
$\sim(A \supset B) \in \Delta$ for any $\Delta \in U(\Gamma \cup \{A\})$	PR.5 (2)
$\mathrm{Pr}(A \supset B, \Gamma) = 0$	Theorem 4.23

POS.12 If, for any Δ, $\mathrm{Pr}(A, \Gamma \cup \Delta) = 1$,
then for any B and C, $\mathrm{Pr}(C, \Gamma \cup \Delta \cup \{B\}) = \mathrm{Pr}(C, \Gamma \cup \Delta \cup \{(A \supset B)\})$

It is a trivial consequence of $L\supset$.

POS.11 If Γ is Pr-normal, then $\mathrm{Pr}(F, \Gamma) = 0$.

It follows from Ax. 3 that $\sim F \in \Delta$ for any $\Delta \in U(\Gamma)$.

POS.13 $\mathrm{Pr}(C, \Gamma \cup \{A_i\}) = 1$ for any i such that $1 \leq i \leq n$, then $\mathrm{Pr}(C, \Gamma \cup \{\bigvee_{i=1}^{n} A_i\}) = 1$.

It is a straightforward consequence of $L\vee$ applies $(n-1)$ times.

POS.15 If $\mathrm{Pr}(C, \Gamma \cup \{\sim A_i\}) = 1$ for any i such that $1 \leq i \leq n$, then $\mathrm{Pr}(C, \Gamma \cup \{\sim(\bigwedge_{i=1}^{n} A_i\})) = 1$.

It is a straightforward consequence of $L\sim\wedge$ applies $(n-1)$ times.

POS.14 If $\mathrm{Pr}(C, \Gamma \cup \{\sim A_1, \ldots, \sim A_n\}) = 1$, then $\mathrm{Pr}(C, \Gamma \cup \{\sim(\bigvee_{i=1}^{n} A_i\})) = 1$

It is a straightforward consequence of $L\sim\vee$ applies $(n-1)$ times.

POS.16

$\mathrm{Pr}(\forall x A, \Gamma) = \lim_{n \to \infty} \mathrm{Pr}(\bigwedge_{i=1}^{n} A[t_i|x], \Gamma)$ where $t_1, \ldots, t_n, \ldots$ is an enumeration of all the terms free for x in A.

There are two cases.

(1)

$\mathrm{Pr}(\forall x A, \Gamma) = 1$	Assumption
For all $\Delta \in U(\Gamma)$, $\forall x A \in \Delta$	Theorem 4.23

$A[t_i\|x], \Delta \Rightarrow A[t_i\|x]$	axiom A1
$A[t_i\|x] \in \Delta$ for all t_i free for x in A	$L\,\forall$ and closure of Δ
$\Pr(\bigwedge_{i=1}^{n} A[t_i\|x], \Gamma) \in \Delta$	$R\wedge$ (n-1) times
$\Pr(\bigwedge_{i=1}^{n} A[t_i\|x], \Gamma) = 1$	Theorem 4.23
$\lim_{n\to\infty} \Pr(\bigwedge_{i=1}^{n} A[t_i\|x], \Gamma) = 1$	calculus

(2)

$\Pr(\forall xA, \Gamma) = 0$	Assumption
For all $\Delta \in U(\Gamma)$, $\sim\forall xA \in \Delta$	Theorem 4.23
$\sim A[y_i\|x], \Delta \Rightarrow \ \sim A[y_i\|x]$	axiom A1
$\sim A[y_i\|x] \in \Delta$ for y_i not free in A, $\sim\forall xA$ and Δ	$R{\sim}\forall$ and the closure of Δ
$\Pr(A[y_i\|x], \Gamma) = 0$	Theorem 4.23
$\Pr(A[t_j\|x], \Gamma) = 0$	for $t_j = y_i$
$\Pr(\bigwedge_{i=1}^{j} A[t_i\|x], \Gamma) = 0$	validity of DF.1
$\lim_{n\to\infty} \Pr(\bigwedge_{i=1}^{n} A[t_i\|x], \Gamma) = 0$	calculus + validity of DF.1

POS.17

$\Pr(\exists xA, \Gamma) = \lim_{n\to\infty} \Pr(\bigvee_{i=1}^{n} A[y_i|x], \Gamma)$ where $y_1, \ldots, y_n, \ldots$ is an enumeration of all the variables that are not free in A and Γ.

The proof is quite similar to that of POS.16 and is left to the reader.

POS.18 If $\Pr(C, \Gamma \cup \{A[t|x]\}) = 1$, then $\Pr(C, \Gamma \cup \{\forall xA\}) = 1$ where t is free for x in A.

We show that $U(\Gamma \cup \{\forall xA\}) \subseteq U(\Gamma \cup \{A[t|x]\})$

$\Delta \in U(\Gamma \cup \{\forall xA\})$	Assumption
$A[t\|x], \Delta \Rightarrow A[t\|x]$	axiom A1
$\forall xA, \Delta \Rightarrow A[t\|x]$	$L\forall$
$A[t\|x] \in \Delta$	closure of Δ
$U(\Gamma \cup \{\forall xA\}) \subseteq U(\Gamma \cup \{A[t\|x]\} \cup \{\forall xA\})$	set theory
$\Pr(C, \Gamma \cup \{\forall xA\}) = \Pr(C, \Gamma \cup \{\forall xA\} \cup \{A[t\|x]\}) = 1$	Theorem 4.10 $+\Pr(C, \Gamma \cup \{A[t\|x]\}) = 1$

POS.19

If $\Pr(C, \Gamma \cup \{A[y|x]\}) = 1$, then $\Pr(C, \Gamma \cup \{\exists x A\}) = 1$ where y is not free in A, Γ and C.

The proof is similar that of POS.18 and is left to the reader.

POS.20 If $\Pr(A[y|x], \Gamma) = 1$ with y not free in Γ nor in A (or $y = x$), then $\Pr(A[t|x], \Gamma) = 1$ where t is free for x in A.

$\Pr(A[y\|x], \Gamma) = 1$	Assumption
$A[y\|x] \in \Delta$ for all $\Delta \in U(\Gamma)$	Theorem 4.23
$\forall x A \in \Delta$ for all $\Delta \in U(\Gamma)$	$R\,\forall$
$A[t\|x], \Gamma \Rightarrow A[t\|x]$	A1
$\forall x A, \Gamma \Rightarrow A[t\|x]$	$L\,\forall$
$A[t\|x] \in \Delta$ for all $\Delta \in U(\Gamma)$	Closure of Δ
$\Pr(A[t\|x], \Gamma) = 1$	Theorem 4.23

□

Theorem 4.28. *SCILSN is complete according to the partial probabilistic interpretation.*

Proof. Let us suppose that $\Gamma \not\Rightarrow A$. There is a $\Delta \in U(\Gamma)$ such that $A \notin \Delta$. Thus $\Pr(A, \Gamma) \neq 1$.

□

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Planar Heyting Algebras for Children

Eduardo Ochs

Abstract

This paper shows a way to interpret (propositional) intuitionistic logic *visually* using finite Planar Heyting Algebras ("ZHAs"), that are certain subsets of $\mathbb{Z}^2$. We also show the connection between ZHAs and the familiar semantics for IPL where the truth-values are open sets: the points of a ZHA H correspond to the open sets of a finite topological space $(P, \mathcal{O}_A(P))$, where the topology $\mathcal{O}_A(P)$ is the order topology on a 2-column graph (P, A). The logic of ZHAs is between classical and intuitionistic but different from both; there are some sentences that are intuitionistically false but that can't have countermodels in ZHAs — their countermodels would need three "columns" or more.

In a wider context these ZHAs are interesting because toposes of the form $\mathbf{Set}^{(P,A)}$ are one of the basic tools for doing "Topos Theory for Children", in the following sense. We can *define* "children" as people who think mathematically in a certain way — as *people who prefer to start from particular cases and finite examples that can be drawn explicitly, and only then generalize* — and we can define a method for working on a particular case (less abstract, "for children") and on a general case ("for adults") in parallel, using parallel diagrams with similar shapes; we have some ways of transfering knowledge from the general case to the particular case, *and back*. This method is sketched in the introduction.

Except for the introduction this paper is self-contained, and its title "Planar Heyting Algebras for Children" also has a second sense, different from the above: it can be read by students who have just taken a basic course on Discrete Mathematics — who are "children" in the sense that they don't have much mathematical maturity — and it prepares these students to read standard books on Logic that they would otherwise find a bit too abstract.

This paper is the first in a series of three. Categories and toposes only appear explicitly in the third one, that is about visualizing geometric morphisms, and at this moment the method of parallel diagrams has only been fully formalized for categorical diagrams. Behind the choices of finite examples and particular cases in this paper there is an *attempt* to adapt that method to areas outside Category Theory, but the precise details of how this is done are left for a future work.

Keywords: Heyting Algebras, Intuitionistic Logic, diagrammatic reasoning.

This paper is the first in a series of three. Let's refer to them as PH1 (this one), PH2 and PH3, and to the whole series as PH123. A nearly complete working draft of PH2 is available at [Och18], and the extended abstract [Och19b] shows the core results that will be in PH3.

The objective of the series can be explained in two ways. In the first one — shallow, and purely mathematical,

- PH1 shows how to interpret IPL in Planar Heyting Algebras ("ZHAs", sec.4) and shows that ZHAs are order topologies on two-column graphs ("2CGs", sec.14); this is used to show how one can develop visual intuition about IPL. The trickiest part is the implication; the method that allows one to calculate $P \to Q$ by sight in ZHAs has four subcases, and is discussed in sections 7, 8, and 9. It would probably be obvious to anyone who has worked enough with lattices, but I believe that it deserves to be more widely known.
- The paper PH2 extends the correspondence $(P, A) \leftrightsquigarrow H$ between 2CGs and ZHAs of PH1 to a correspondence $((P, A), Q) \leftrightsquigarrow (H, J)$ between 2CGs "with question marks" and ZHAs with a J-operator — that, more visually, are ZHAs "with slashings".
- PH3 transports this to Topos Theory: if we regard a 2CG (P, A) as a category, then $\mathbf{Set}^{(P,A)}$ is a topos whose objects are easy to draw, and the logic of $\mathbf{Set}^{(P,A)}$ is exactly the ZHA associated to (P, A); also, a set of question marks $Q \subseteq P$ induces an operation on $\mathbf{Set}^{(P,A)}$ that erases the information on Q and reconstructs it in a natural way, and this erasing-plus-reconstruction yields a sheafification functor — that is exactly the one associated to the local operator j associated to the J-operator J. This gives us a way to visualize (certain) toposes, sheaves, geometric morphisms, and two factorizations of geometric morphisms.

The second way to explain the goals of PH123 is by taking *Diagrammatic Reasoning* as the main theme. Let me start with an anecdote (90% true). Many, many years ago, when I tried to learn Topos Theory for the first time, mainly from [Joh77] and [Gol84], everything felt far too abstract: most of the diagrams were omitted, and the motivating examples were mentioned very briefly, if at all. The intended audience for those books surely knew how to supply by themselves the missing diagrams, examples, calculations, and details — but I didn't. My slogan became: "I need a version for children of this!".

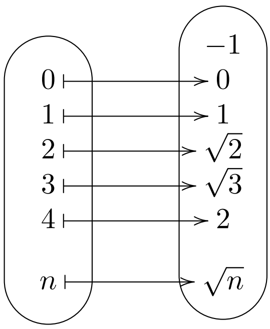
At first this expression, "for children", was informal, and I used it as a half-joke. Very gradually it started to acquire a precise sense: clearly, CT done in a purely algebraic way is "for adults", and diagrams, particular cases, and finite examples are "for children". Writing "for adults" only and keeping the mentions to the "for children" part very brief is considered good style because "adults" have the technical machinery for producing more or less automatically the "for children" part when they need it, and people who are not yet "adults" can become "adults" by struggling with the texts "for adults" long enough and learning by themselves how to handle the new level of abstraction.

A clear frontier between "for adults" and "for children" appears when we realize that we can draw a diagram for the general case ("for adults") of a categorical concept and the diagram for a particular case of it ("for children") side by side. The two diagrams will have roughly the same shape, and we can transport knowledge between them in both ways: from the general to the particular, *and back*. Look at Figure 1; let's name its subdiagrams as A, B, and C, like this: ${}^{A\,B}_{\ C}$. Each one of A, B, C has an *internal view* above and an *external view* below.

Diagram A shows, below, the external view of the function $\mathbb{N} \overset{\sqrt{\ }}{\to} \mathbb{R}$, and above that its internal view — in which one of the arrows, $n \mapsto \sqrt{n}$, shows the action of $\sqrt{\ }$ on a generic element, and the other '$\mapsto$' arrows, like $3 \mapsto \sqrt{3}$ and $4 \mapsto 2$, show substitution instances of $n \mapsto \sqrt{n}$, maybe after some term reductions.

Diagram B shows the external view of a (generic) adjunction $L \dashv R$, and above it its internal view. The nodes and arrows above $\mathbf{B}$ are objects and morphisms in $\mathbf{B}$, and similarly for the nodes and arrows above $\mathbf{A}$. The '$\mapsto$' arrows of the internal view are now of three kinds: actions of functors on objects, actions of functors on morphisms, and "transpositions" coming from the natural isomorphism $\mathrm{Hom}(L-,-) \leftrightarrow \mathrm{Hom}(-,R-)$. Diagram C is essentially the same as B, but for a particular adjunction: $(\times B) \dashv (B{\to})$. Note how the diagrams B and C have exactly the same shape — but our diagrams for internal views are much bigger than the corresponding external views.

For a case in which the interplay between external and internal views is examined in full detail, see [Och19a]; it shows how each node and arrow in the diagrams can be can interpreted as a term in a type system, and this *may* be a basis for analyzing precisely which kinds of knowledge, and which kinds of intuitions — as in [Krö07], especially sec.1.3.2, and in [Cor04] — we are transporting from the less abstract diagrams to the more abstract ones, and vice-versa. Note that having a clearly-defined method for lifting information — in the sense of [Och13] — from a case "for children" to a case "for adults" would allow people to publish much more material "for children" than they do now, without this being regarded as bad style. For a non-trivial example of



$$\mathbb{N} \xrightarrow{\sqrt{\ }} \mathbb{R}$$

$$\begin{tikzcd}
 & LA' \arrow[d, "L\alpha"'] & A' \arrow[l, mapsto, "L_0"'] \arrow[d, "\alpha"] \arrow[dl, phantom, "\xleftarrow{L_1}"] & \\
LRB \arrow[d, "\epsilon B"'] & LA \arrow[d, "\substack{g^\flat \\ f}"'] & A \arrow[l, mapsto, "L_0"'] \arrow[d, "\substack{g \\ f^\sharp}"] \arrow[dl, phantom, "\underset{\sharp}{\overset{\flat}{\rightleftarrows}}"] & A \arrow[d, "\eta A"] \\
B & B \arrow[r, mapsto, "R_0"'] \arrow[d, "\beta"'] \arrow[dr, phantom, "\xmapsto[R_1]{}"] & RB \arrow[d, "R\beta"] & RLA \\
 & B' \arrow[r, mapsto, "R_0"'] & RB' &
\end{tikzcd}$$

$$\mathbf{B} \underset{R}{\overset{L}{\leftrightarrows}} \mathbf{A}$$

$$\begin{tikzcd}
 & A{\times}C \arrow[d, "\lambda p.(f(\pi p){,}\pi' p)"'] & A \arrow[l, mapsto] \arrow[d, "f"] \arrow[dl, phantom, "\leftarrow\!\shortmid"] & \\
(C{\to}D){\times}C \arrow[d, "\lambda p.(\pi p)(\pi' p)"'] & B{\times}C \arrow[d, "\substack{\lambda p.g(\pi p)(\pi' p) \\ h}"'] & B \arrow[l, mapsto] \arrow[d, "\substack{g \\ \lambda b.\lambda c.h(b{,}c)}"] \arrow[dl, phantom, "\underset{\sharp}{\overset{\flat}{\rightleftarrows}}"] & B \arrow[d, "\lambda b.\lambda c.(b{,}c)"] \\
D & D \arrow[r, mapsto] \arrow[d, "k"'] \arrow[dr, phantom, "\mapsto"] & (C{\to}D) \arrow[d, "\lambda f.\lambda c.k(fc)"] & (C{\to}(B{\times}C)) \\
 & E \arrow[r, mapsto] & (C{\to}E) &
\end{tikzcd}$$

$$\mathbf{Set} \underset{(C\to)}{\overset{(\times C)}{\leftrightarrows}} \mathbf{Set}$$

Figure 1: Three cases of internal views and external views.

lifting information from a particular case to a general case, see [Och19b].

This paper can be seen as part of bigger projects in at least the two ways described above, but it was also written to be as readable and as self-contained as possible. In 2016 and 2017 I had the opportunity to test some of the ideas here on "real children", in the sense of "people with little mathematical *knowledge* and little mathetical *maturity*". I gave a seminar course about Logic and λ-calculus that had no prerequisites, and that was mostly based on exercises that the students would try to solve together by discussing on the whiteboard; it was mostly attended by Computer Science students who had just finished a course on Discrete Mathematics, but there were also some Psychology and Art students — that unfortunately left after the first weeks of each semester. All these students, including the CompSci ones, had in common that definitions only made sense to them after they had played with a few concrete examples; at some parts of the course I would ask them to read some sections of this paper, then work on some extra exercises that I had prepared, and then read excerpts of books like [Dal08] or [Awo06]. Most sections of this paper had been tested "on real children" in this way, and were rewritten several times after their feedback and reactions. I owe them many thanks — I'm glad that they had fun in the process – and I hope that I'll be able in the future to transform what I learned with them into precise techniques for writing "for children".

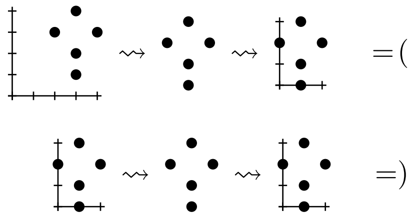
1 Positional notations

Definition: a *ZSet* is a finite, non-empty subset of $\mathbb{N}^2$ that touches both axes, i.e., that has a point of the form $(0, \_)$ and a point of the form $(\_, 0)$. We will often represent ZSets using a bullet notation, with or without the axes and ticks. For example:

$$K = \left\{ \begin{array}{ccc} & (1,3), & \\ (0,2), & & (2,2), \\ & (1,1), & \\ & (1,0) & \end{array} \right\} = \begin{array}{|ccc} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \\ \hline & \bullet & \end{array} = \begin{array}{ccc} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \\ & \bullet & \end{array}$$

We will use the ZSet above a lot in examples, so let's give it a short name: K ("kite").

The condition of touching both axes is what lets us represent ZSets unambiguously using just the bullets:



We can use a positional notation to represent functions *from* a ZSet. For example, if

$$\begin{array}{rcrcl} f & : & K & \to & \mathbb{N} \\ & & (x,y) & \mapsto & x \end{array}$$

then

$$f = \left\{ \begin{smallmatrix} & ((1,3),1), & \\ ((0,2),0), & & ((2,2),2), \\ & ((1,1),1), & \\ & ((1,0),1) & \end{smallmatrix} \right\} = \begin{smallmatrix} & 1 & \\ 0 & & 2 \\ & 1 & \\ & 1 & \end{smallmatrix}$$

We will sometimes use λ-notation to represent functions compactly. For example:

$$\lambda(x,y){:}K.x = \left\{ \begin{smallmatrix} & ((1,3),1), & \\ ((0,2),0), & & ((2,2),2), \\ & ((1,1),1), & \\ & ((1,0),1) & \end{smallmatrix} \right\} = \begin{smallmatrix} & 1 & \\ 0 & & 2 \\ & 1 & \\ & 1 & \end{smallmatrix}$$

$$\lambda(x,y){:}K.y = \left\{ \begin{smallmatrix} & ((1,3),3), & \\ ((0,2),2), & & ((2,2),2), \\ & ((1,1),1), & \\ & ((1,0),0) & \end{smallmatrix} \right\} = \begin{smallmatrix} & 3 & \\ 2 & & 2 \\ & 1 & \\ & 0 & \end{smallmatrix}$$

The "reading order" on the points of a ZSet S "lists" the points of S starting from the top and going from left to right in each line. More precisely, if S has n points then $r_S : S \to \{1, \dots, n\}$ is a bijection, and for example:

$$r_K = \begin{smallmatrix} & 1 & \\ 2 & & 3 \\ & 4 & \\ & 5 & \end{smallmatrix}$$

Subsets of a ZSet are represented with a notation with '$\bullet$'s and '$\cdot$', and partial functions from a ZSet are represented with '$\cdot$'s where they are not defined. For example:

$$\begin{smallmatrix} & \bullet & \\ \cdot & & \bullet \\ & \bullet & \\ & \cdot & \end{smallmatrix} \qquad \begin{smallmatrix} & 1 & \\ \cdot & & 3 \\ & 4 & \\ & \cdot & \end{smallmatrix}$$

The *characteristic function* of a subset S' of a ZSet S is the function $\chi_{S'} : S \to \{0,1\}$ that returns 1 exactly on the points of S'; for example, $\begin{smallmatrix} & 1 & \\ 0 & & 1 \\ & 1 & \\ & 0 & \end{smallmatrix}$ is the characteristic function of $\begin{smallmatrix} & \bullet & \\ \cdot & & \bullet \\ & \bullet & \\ & \cdot & \end{smallmatrix} \subset \begin{smallmatrix} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \\ & \bullet & \end{smallmatrix}$. We will sometimes denote subsets by their characteristic functions because this makes them easier to “pronounce” by reading aloud their digits in the reading order — for example, $\substack{1\\0\;\;\,1\\1\\0}$ is “one-zero-one-one-zero” (see sec.12).

2 ZDAGs

We will sometimes use the bullet notation for a ZSet S as a *shorthand* for one of the two DAGs induced by S: one with its arrows going up, the other one with them going down. For example: sometimes

$$
\begin{array}{ccc} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \\ & \bullet & \end{array}
\quad \text{will stand for:} \quad
\begin{array}{ccccc} & & \bullet & & \\ & \swarrow & & \searrow & \\ \bullet & & & & \bullet \\ & \searrow & & \swarrow & \\ & & \bullet & & \\ & & \downarrow & & \\ & & \bullet & & \end{array}
=
\begin{array}{ccccc} & & (1,3) & & \\ & \swarrow & & \searrow & \\ (0,2) & & & & (2,2) \\ & \searrow & & \swarrow & \\ & & (1,1) & & \\ & & \downarrow & & \\ & & (1,0) & & \end{array}
=
$$

$$
\left(\left\{ \begin{array}{ccc} & (1,3), & \\ (0,2), & & (2,2), \\ & (1,1), & \\ & (0,0) & \end{array} \right\}, \left\{ \begin{array}{c} ((1,3),(0,2)),((1,3),(2,2)), \\ ((0,2),(1,1)),((2,2),(1,1)), \\ ((1,1),(0,0)) \end{array} \right\} \right)
$$

Let’s formalize this.

Consider a game in which black and white pawns are placed on points of $\mathbb{Z}^2$, and they can move like this:

$$
\begin{array}{ccc} & \bullet & \\ \swarrow & \downarrow & \searrow \\ \bullet & \bullet & \bullet \end{array}
\qquad
\begin{array}{ccc} \circ & \circ & \circ \\ \nwarrow & \uparrow & \nearrow \\ & \circ & \end{array}
$$

Black pawns can move from (x, y) to $(x + k, y - 1)$ and white pawns from (x, y) to $(x + k, y + 1)$, where $k \in \{-1, 0, 1\}$. The mnemonic is that black pawns are “solid”, and thus “heavy”, and they “sink”, so they move down; white pawns are “hollow”, and thus “light”, and they “float”, so they move up.

Let’s now restrict the board positions to a ZSet S. Black pawns can move from (x, y) to $(x+k, y-1)$ and white pawns from (x, y) to $(x+k, y+1)$, where $k \in \{-1, 0, 1\}$, but only when the starting and ending positions both belong to S. The sets of possible black pawn moves and white pawn moves on S can be defined formally as:

$$
\begin{aligned}
\mathsf{BPM}(S) &= \{\, ((x, y), (x', y')) \in S^2 \mid x - x' \in \{-1, 0, 1\}, y' = y - 1 \,\} \\
\mathsf{WPM}(S) &= \{\, ((x, y), (x', y')) \in S^2 \mid x - x' \in \{-1, 0, 1\}, y' = y + 1 \,\}
\end{aligned}
$$

...and now please forget everything else you expect from a game — like starting position, capturing, objective, winning... the idea of a "game" was just a tool to let us explain $\mathsf{BPM}(S)$ and $\mathsf{WPM}(S)$ quickly.

A *ZDAG* is a DAG of the form $(S, \mathsf{BPM}(S))$ or $(S, \mathsf{WPM}(S))$, where S is a ZSet.

A *ZPO* is partial order of the form $(S, \mathsf{BPM}(S)^*)$ or $(S, \mathsf{WPM}(S)^*)$, where S is a ZSet and the '$*$' denotes the transitive-reflexive closure of the relation.

Sometimes, when this is clear from the context, a bullet diagram like $\begin{smallmatrix} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \\ & \bullet & \end{smallmatrix}$ will stand for either the ZDAGs $(\begin{smallmatrix} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \\ & \bullet & \end{smallmatrix}, \mathsf{BPM}(\begin{smallmatrix} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \\ & \bullet & \end{smallmatrix}))$ or $(\begin{smallmatrix} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \\ & \bullet & \end{smallmatrix}, \mathsf{WPM}(\begin{smallmatrix} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \\ & \bullet & \end{smallmatrix}))$, or for the ZPOs $(\begin{smallmatrix} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \\ & \bullet & \end{smallmatrix}, \mathsf{BPM}(\begin{smallmatrix} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \\ & \bullet & \end{smallmatrix})^*)$ or $(\begin{smallmatrix} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \\ & \bullet & \end{smallmatrix}, \mathsf{WPM}(\begin{smallmatrix} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \\ & \bullet & \end{smallmatrix})^*)$ (sec.4).

3 LR-coordinates

The *lr-coordinates* are useful for working on quarter-plane of $\mathbb{Z}^2$ that looks like $\mathbb{N}^2$ turned 45° to the left. Let $\langle l, r \rangle := (-l + r, l + r)$; then (the bottom part of) $\{ \langle l, r \rangle \mid l, r \in \mathbb{N} \}$ is:

$$\begin{array}{ccccccccc}
\langle 4,0\rangle & & \langle 3,1\rangle & & \langle 2,2\rangle & & \langle 1,3\rangle & & \langle 0,4\rangle \\
& \langle 3,0\rangle & & \langle 2,1\rangle & & \langle 1,2\rangle & & \langle 0,3\rangle & \\
& & \langle 2,0\rangle & & \langle 1,1\rangle & & \langle 0,2\rangle & & \\
& & & \langle 1,0\rangle & & \langle 0,1\rangle & & & \\
& & & & \langle 0,0\rangle & & & &
\end{array}
\quad = \quad
\begin{array}{ccccccccc}
(-4,4) & & (-2,4) & & (0,4) & & (2,4) & & (4,4) \\
& (-3,3) & & (-1,3) & & (1,3) & & (3,3) & \\
& & (-2,2) & & (0,2) & & (2,2) & & \\
& & & (-1,1) & & (1,1) & & & \\
& & & & (0,0) & & & &
\end{array}$$

Sometimes we will write lr instead of $\langle l, r \rangle$. So:

$$\begin{array}{ccccccccc}
40 & & 31 & & 22 & & 13 & & 04 \\
& 30 & & 21 & & 12 & & 03 & \\
& & 20 & & 11 & & 02 & & \\
& & & 10 & & 01 & & & \\
& & & & 00 & & & &
\end{array}
\quad = \quad
\begin{array}{ccccccccc}
(-4,4) & & (-2,4) & & (0,4) & & (2,4) & & (4,4) \\
& (-3,3) & & (-1,3) & & (1,3) & & (3,3) & \\
& & (-2,2) & & (0,2) & & (2,2) & & \\
& & & (-1,1) & & (1,1) & & & \\
& & & & (0,0) & & & &
\end{array}$$

Let $\mathbb{LR} = \{ \langle l, r \rangle \mid l, r \in \mathbb{N} \}$.

4 ZHAs

A *ZHA* is a subset of $\mathbb{LR}$ "between a left and a right wall", as we will see.

A triple (h, L, R) is a “height-left-right-wall” when:
1) $h \in \mathbb{N}$
2) $L : \{0, \ldots, h\} \to \mathbb{Z}$ and $R : \{0, \ldots, h\} \to \mathbb{Z}$
3) $L(h) = R(h)$ (the top points of the walls are the same)
4) $L(0) = R(0) = 0$ (the bottom points of the walls are the same, 0)
5) $\forall y \in \{0, \ldots, h\}.\, L(y) \leq R(y)$ (“left” is left of “right”)
6) $\forall y \in \{1, \ldots, h\}.\, L(y) - L(y-1) = \pm 1$ (the left wall makes no jumps)
7) $\forall y \in \{1, \ldots, h\}.\, R(y) - R(y-1) = \pm 1$ (the right wall makes no jumps)

The *ZHA generated by* a height-left-right-wall (h, L, R) is the set of all points of $\mathbb{LR}$ with valid height and between the left and the right walls. Formally:

$$\mathsf{ZHAG}(h, L, R) = \{\, (x, y) \in \mathbb{LR} \mid y \leq h, L(y) \leq x \leq R(y) \,\}.$$

A *ZHA* is a set of the form $\mathsf{ZHAG}(h, L, R)$, where the triple (h, L, R) is a height-left-right-wall.

Here is an example of a ZHA (with the white pawn moves on it):

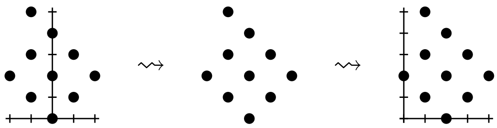
$$\begin{array}{ccccccccccc}
 & & (-3,9) & & & & & & & & \\
 & \nearrow & & \nwarrow & & & & & & & \\
(-4,8) & & & & (-2,8) & & & & & & \\
 & \nwarrow & & \nearrow & & & & & & & \\
 & & (-3,7) & & & & & & & & \\
 & & & \nwarrow & & & & & & & \\
 & & & & (-2,6) & & & & & & \\
 & & & & & \nwarrow & & & & & \\
 & & & & & & (-1,5) & & & & \\
 & & & & & \nearrow & & \nwarrow & & & \\
 & & & & (-2,4) & & & & (0,4) & & \\
 & & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & \\
 & & (-3,3) & & & & (-1,3) & & & & (1,3) \\
 & & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & \\
 & & & & (-2,2) & & & & (0,2) & & \\
 & & & & & \nwarrow & & \nearrow & & \nwarrow & \\
 & & & & & & (-1,1) & & & & (1,1) \\
 & & & & & & & \nwarrow & & \nearrow & \\
 & & & & & & & & (0,0) & &
\end{array}$$

$L(9) = -3$	$R(9) = -3$	$L(9) = R(9)$	$h = 9$
$L(8) = -4$	$R(8) = -2$		
$L(7) = -3$	$R(7) = -3$		
$L(6) = -2$	$R(6) = -2$		
$L(5) = -1$	$R(5) = -1$		
$L(4) = -2$	$R(4) = 0$		
$L(3) = -3$	$R(3) = 1$		
$L(2) = -2$	$R(2) = 0$		
$L(1) = -1$	$R(1) = 1$		
$L(0) = 0$	$R(0) = 0$	$L(0) = R(0) = 0$	

We will see later (in section 7) that ZHAs (with white pawn moves) are Heyting Algebras.

5 Conventions on diagrams without axes

We can use a bullet notation to denote ZHAs, but look at what happens when we start with a ZHA, erase the axes, and then add the axes back using the convention from sec.1:



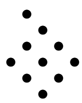
The new, restored axes are in a different position — the bottom point of the original ZHA at the left was $(0,0)$, but in the ZSet at the right the bottom point is $(2,0)$.

The convention from sec.1 is not adequate for ZHAs.

Let's modify it!

From this point on, the convention on where to draw the axes will be this one: *when it is clear from the context that a bullet diagram represents a ZHA,* then its (unique) bottom point has coordinate $(0,0)$, and we use that to draw the axes; otherwise we apply the old convention, that chooses $(0,0)$ as the point that makes the diagram fit in $\mathbb{N}^2$ and touch both axes.

The new convention with two cases also applies to functions from ZHAs, and to partial functions and subsets. For example:

$B =$  (a ZHA) $\qquad \lambda(x,y){:}B.x = \begin{smallmatrix} & \text{-1} & & & \\ & & 0 & & \\ & \text{-1} & & 1 & \\ \text{-2} & & 0 & & 2 \\ & \text{-1} & & 1 & \\ & & 0 & & \end{smallmatrix}$

$$\lambda\langle l,r\rangle{:}B.l = \begin{smallmatrix} & 3 & & & \\ & & 2 & & \\ & 2 & & 1 & \\ 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{smallmatrix} \qquad \lambda\langle l,r\rangle{:}B.r = \begin{smallmatrix} & 2 & & & \\ & & 2 & & \\ & 1 & & 2 & \\ 0 & & 1 & & 2 \\ & 0 & & 1 & \\ & & 0 & & \end{smallmatrix}$$

We will often denote ZHAs by the identity function on them:

$$\lambda\langle l,r\rangle{:}B.\langle l,r\rangle \;=\; \lambda lr{:}B.lr \;=\; \begin{smallmatrix} & 32 & & & \\ & & 22 & & \\ & 21 & & 12 & \\ 20 & & 11 & & 02 \\ & 10 & & 01 & \\ & & 00 & & \end{smallmatrix} \qquad B \;=\; \begin{smallmatrix} & 32 & & & \\ & & 22 & & \\ & 21 & & 12 & \\ 20 & & 11 & & 02 \\ & 10 & & 01 & \\ & & 00 & & \end{smallmatrix}$$

Note that we are using the compact notation from the end of section 3: 'lr' instead of '$\langle l,r\rangle$'.

6 Propositional calculus

A *PC-structure* is a tuple

$$L = (\Omega, \leq, \top, \bot, \wedge, \vee, \rightarrow, \leftrightarrow, \neg),$$

where:

Ω is the "set of truth values",
$\leq$ is a relation on Ω,
$\top$ and $\bot$ are two elements of Ω,
$\wedge$, $\vee$, $\to$, $\leftrightarrow$ are functions from $\Omega \times \Omega$ to Ω,
$\neg$ is a function from Ω to Ω.

Classical Logic "is" a PC-structure, with $\Omega = \{0,1\}$, $\top = 1$, $\bot = 0$, $\leq = \{(0,0),(0,1),(1,0)\}$, $\wedge = \left\{ \begin{smallmatrix} ((0,0),0),((0,1),0), \\ ((1,0),0),((1,1),1) \end{smallmatrix} \right\}$, etc.

PC-structures let us interpret expressions from Propositional Calculus ("PC-expressions"), and let us define a notion of *tautology*. For example, in Classical Logic,

- $\neg\neg P \leftrightarrow P$ is a tautology because it is valid (i.e., it yields $\top$) for all values of P in Ω,
- $\neg(P \wedge Q) \to (\neg P \vee \neg Q)$ is a tautology because it is valid for all values of P and Q in Ω,
- but $P \vee Q \to P \wedge Q$ is *not* a tautology, because when $P = 0$ and $Q = 1$ the result is not $\top$:

$$\underbrace{\underbrace{\underbrace{P}_{0} \vee \underbrace{Q}_{1}}_{1} \to \underbrace{\underbrace{P}_{0} \wedge \underbrace{Q}_{1}}_{0}}_{0}$$

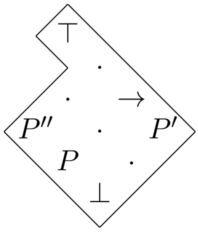
7 Propositional calculus in a ZHA

Let Ω be the set of points of a ZHA and $\leq$ the default partial order on it. The *default meanings for* $\top, \bot, \land, \lor, \to, \leftrightarrow, \neg$ are these ones:

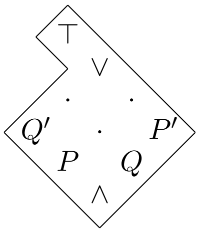
$$
\begin{array}{rcl}
\langle a,b\rangle \leq \langle c,d\rangle &:=& a \leq c \land b \leq d \\
\langle a,b\rangle \geq \langle c,d\rangle &:=& a \geq c \land b \geq d \\[4pt]
\langle a,b\rangle \ \mathsf{above}\ \langle c,d\rangle &:=& a \geq c \land b \geq d \\
\langle a,b\rangle \ \mathsf{below}\ \langle c,d\rangle &:=& a \leq c \land b \leq d \\
\langle a,b\rangle \ \mathsf{leftof}\ \langle c,d\rangle &:=& a \geq c \land b \leq d \\
\langle a,b\rangle \ \mathsf{rightof}\ \langle c,d\rangle &:=& a \leq c \land b \geq d \\[4pt]
\mathsf{valid}(\langle a,b\rangle) &:=& \langle a,b\rangle \in \Omega \\
\mathsf{ne}(\langle a,b\rangle) &:=& \mathsf{if\ valid}\ (\langle a,b+1\rangle)\ \mathsf{then\ ne}(\langle a,b+1\rangle)\ \mathsf{else}\ \langle a,b\rangle\ \mathsf{end} \\
\mathsf{nw}(\langle a,b\rangle) &:=& \mathsf{if\ valid}\ (\langle a+1,b\rangle)\ \mathsf{then\ nw}(\langle a+1,b\rangle)\ \mathsf{else}\ \langle a,b\rangle\ \mathsf{end} \\[4pt]
\langle a,b\rangle \land \langle c,d\rangle &:=& \langle \mathsf{min}(a,c), \mathsf{min}(b,d)\rangle \\
\langle a,b\rangle \lor \langle c,d\rangle &:=& \langle \mathsf{max}(a,c), \mathsf{max}(b,d)\rangle \\[4pt]
\langle a,b\rangle \to \langle c,d\rangle &:=& \begin{array}{llll}
\mathsf{if} & \langle a,b\rangle\ \mathsf{below}\ \langle c,d\rangle & \mathsf{then} & \top \\
\mathsf{elseif} & \langle a,b\rangle\ \mathsf{leftof}\ \langle c,d\rangle & \mathsf{then} & \mathsf{ne}(\langle c,d\rangle) \\
\mathsf{elseif} & \langle a,b\rangle\ \mathsf{rightof}\ \langle c,d\rangle & \mathsf{then} & \mathsf{nw}(\langle c,d\rangle) \\
\mathsf{elseif} & \langle a,b\rangle\ \mathsf{above}\ \langle c,d\rangle & \mathsf{then} & \langle c,d\rangle \\
\mathsf{end} & & &
\end{array} \\
\top &:=& \mathsf{sup}(\Omega) \\
\bot &:=& \langle 0,0\rangle \\
\neg\langle a,b\rangle &:=& \langle a,b\rangle \to \bot \\
\langle a,b\rangle \leftrightarrow \langle c,d\rangle &:=& (\langle a,b\rangle \to \langle c,d\rangle) \land (\langle c,d\rangle \to \langle a,b\rangle)
\end{array}
$$

Let Ω be the ZHA at the top left in the figure below. Then, with the default meanings for the connectives neither $\neg\neg P \to P$ nor $\neg(P \land Q) \to (\neg P \lor \neg Q)$ are tautologies, as there are valuations that make them yield results different than $\top = 32$:

$$\begin{array}{ccccc}
 & 32 & & & \\
 & & 22 & & \\
 & 21 & & 12 & \\
20 & & 11 & & 02 \\
 & 10 & & 01 & \\
 & & 00 & & \\
\end{array}$$



$$\underbrace{(\underbrace{\neg\underbrace{\neg\underbrace{P}_{10}}_{02}}_{20}) \to \underbrace{P}_{10}}_{12}$$



$$\underbrace{\underbrace{\neg(\underbrace{\underbrace{P}_{10} \land \underbrace{Q}_{01}}_{00})}_{32} \to \underbrace{(\underbrace{\neg\underbrace{P}_{10}}_{02} \lor \underbrace{\neg\underbrace{Q}_{01}}_{20})}_{22}}_{22}$$

So: *some* classical tautologies are not tautologies in this ZHA.

The somewhat arbitrary-looking definition of '→' will be explained at the end of the next section.

8 Heyting Algebras

A *Heyting Algebra* is a PC-structure

$$H = (\Omega, \leq_H, \top_H, \bot_H, \land_H, \lor_H, \to_H, \leftrightarrow_H, \neg_H),$$

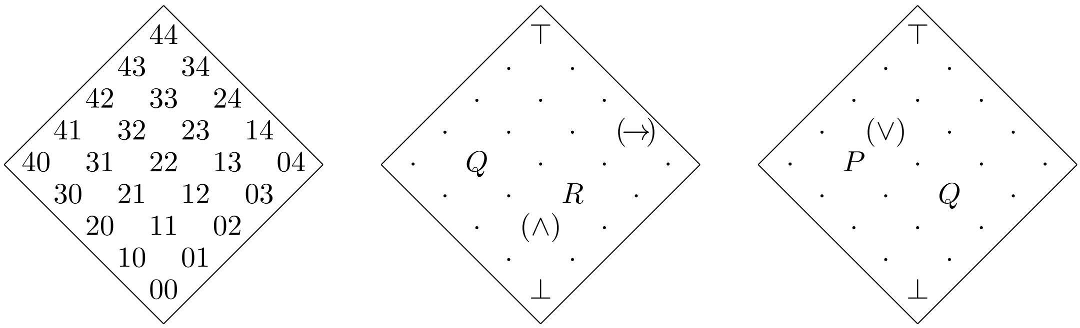
in which:

1) $(\Omega, \leq_H)$ is a partial order
2) $\top_H$ is the top element of the partial order
3) $\bot_H$ is the bottom element of the partial order
4) $P \leftrightarrow_H Q$ is the same as $(P \to_H Q) \land_H (Q \to_H P)$
5) $\neg_H P$ is the same as $P \to_H \bot_H$
6) $\forall P, Q, R \in \Omega.\ (P \leq_H (Q \land_H R)) \leftrightarrow ((P \leq_H Q) \land (P \leq_H R))$
7) $\forall P, Q, R \in \Omega.\ ((P \lor_H Q) \leq_H R) \leftrightarrow ((P \leq_H R) \land (Q \leq_H R))$
8) $\forall P, Q, R \in \Omega.\ (P \leq_H (Q \to_H R)) \leftrightarrow ((P \land_H Q) \leq_H R)$
6') $\forall Q, R \in \Omega. \exists! Y \in \Omega. \forall P \in \Omega.\ (P \leq_H Y) \leftrightarrow ((P \leq_H Q) \land (P \leq_H R))$
7') $\forall P, Q \in \Omega. \exists! X \in \Omega. \forall R \in \Omega.\ (X \leq_H R) \leftrightarrow ((P \leq_H R) \land (Q \leq_H R))$
8') $\forall Q, R \in \Omega. \exists! Y \in \Omega. \forall P \in \Omega.\ (P \leq_H Y) \leftrightarrow ((P \land_H R) \leq_H R)$

The conditions 6', 7', 8' say that there are unique elements in Ω that "behave as" $Q \land_H R$, $P \lor_H Q$ and $Q \to_H R$ for given P, Q, R; the conditions 6,7,8 say that $Q \land_H R$, $P \lor_H Q$ and $Q \to_H R$ are exactly the elements with this behavior.

The positional notation on ZHAs is very helpful for visualizing what the conditions 6',7',8',6,7,8 "mean". More precisely: once we fix a ZHA Ω and truth-values $P, Q, R \in \Omega$ we have a way to draw and to visualize the "behavior" of each subexpression of the conditions 6, 7, 8 using the positional notations of sec.1, and we can use that to obtain the only possible values for $Q \wedge_H R$, $P \vee_H Q$ and $Q \to_H R$.

We will examine three particular cases: with Ω being the ZDAG on the left below,



a) if $Q = 31$ and $R = 12$ then $Q \wedge_H R = 11$,
b) if $P = 31$ and $Q = 12$ then $P \vee_H Q = 32$,
c) if $Q = 31$ and $R = 12$ then $Q \to_H R = 14$.

Before we start, note that in 6, 7, 8, 6', 7', 8' some subexpressions yield truth values in Ω and other subexpressions yield standard truth values. For example, in 6, with $P = 20$, we have:

$$\underbrace{\underbrace{(\underbrace{P}_{20} \leq_H \underbrace{(\underbrace{Q}_{31} \wedge_H \underbrace{R}_{12})}_{11})}_{0} \leftrightarrow \underbrace{(\underbrace{(\underbrace{P}_{20} \leq_H \underbrace{Q}_{31})}_{1} \wedge \underbrace{(\underbrace{P}_{20} \leq_H \underbrace{R}_{12})}_{0})}_{0}}_{1}$$

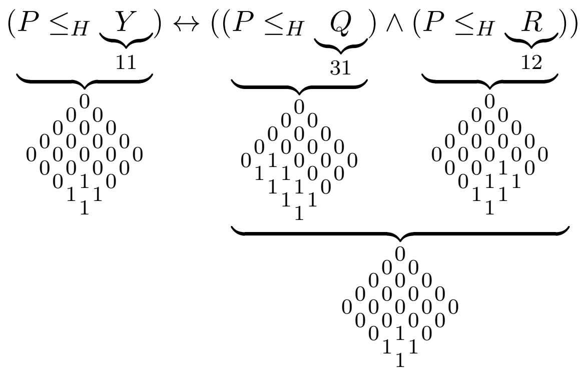
Case (a). Let $Q = 31$ and $R = 12$. We want to see that $Q \wedge_H R = 11$, i.e., that

$$\forall P \in \Omega.\ (P \leq_H Y) \leftrightarrow ((P \leq_H Q) \wedge (P \leq_H R))$$

holds for $Y = 11$ and for no other $Y \in \Omega$. We can visualize the behavior of $P \leq_H Q$ for all 'P's by drawing $\lambda P{:}\Omega.(P \leq_H Q)$ in the positional notation; then we do the same for $\lambda P{:}\Omega.(P \leq_H R)$ and for $\lambda P{:}\Omega.((P \leq_H Q) \wedge (P \leq_H R))$. Suppose that the full expression, '$\forall P{:}\Omega.$ ___', is true; then the behavior of the left side of the '$\leftrightarrow$', $\lambda P{:}\Omega.(P \leq_H Y)$, has to be a copy of the behavior of the right side, and that lets us find the only adequate value for Y.

The order in which we calculate and draw things is below, followed by the results themselves:

$$\underbrace{(P \leq_H \underbrace{Y}_{(7)})}_{(6)} \leftrightarrow (\underbrace{\underbrace{(P \leq_H \underbrace{Q}_{(1)})}_{(3)} \land \underbrace{(P \leq_H \underbrace{R}_{(2)})}_{(4)}}_{(5)})$$

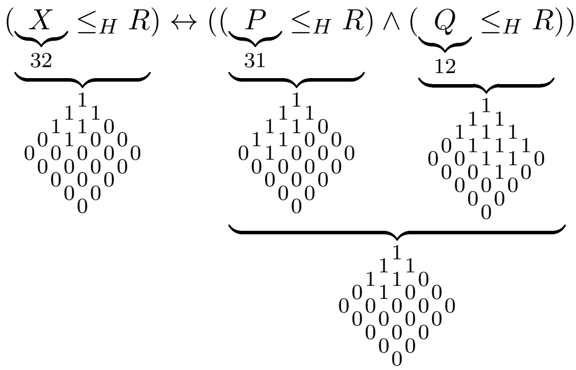


Case (b). Let $P = 31$ and $Q = 12$. We want to see that $P \vee_H Q = 32$, i.e., that

$$\forall R{:}\Omega.\ \ (X \leq_H R) \leftrightarrow ((P \leq_H R) \land (Q \leq_H R))$$

holds for $X = 32$ and for no other $X \in \Omega$. We do essentially the same as we did in (a), but now we calculate $\lambda R{:}\Omega.(P \leq_H R)$, $\lambda R{:}\Omega.(Q \leq_H R)$, and $\lambda R{:}\Omega.((P \leq_H R) \land (Q \leq_H R))$. The order in which we calculate and draw things is below, followed by the results themselves:

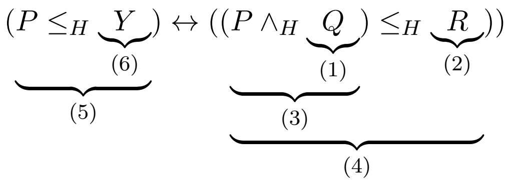
$$\underbrace{(\underbrace{X}_{(7)} \leq_H R)}_{(6)} \leftrightarrow (\underbrace{\underbrace{(\underbrace{P}_{(1)} \leq_H R)}_{(3)} \land \underbrace{(\underbrace{Q}_{(2)} \leq_H R)}_{(4)}}_{(5)})$$

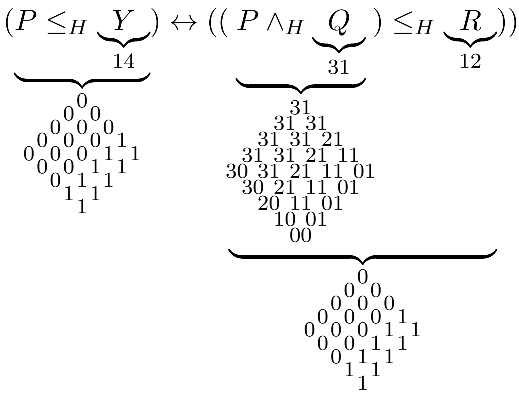


Case (c). Let $Q = 31$ and $R = 12$. We want to see that $Q \to_H R = 14$, i.e., that

$$\forall P{:}\Omega.\ (P \leq_H Y) \leftrightarrow ((P \wedge_H Q) \leq_H R)$$

holds for $Y = 14$ and for no other $Y \in \Omega$. Here we have to do something slightly different. We start by visualizing $\lambda P{:}\Omega.(P \wedge_H Q)$, which is a function from Ω to Ω, not a function from Ω to $\{0, 1\}$ like the ones we were using before. The order in which we calculate and draw things is below, followed by the results:





9 The two implications are equivalent

In sec.7 we gave a definition of '$\to$' that is easy to calculate, and in sec.8 we saw a way to find by brute force[1] a value for $Q \to R$ that obeys

$$(P \leq (Q \to R)) \leftrightarrow (P \wedge Q \leq R)$$

for all P. In this section we will see a proof that these two operations — called '$\stackrel{\mathrm{C}}{\to}$' and '$\stackrel{\mathrm{HA}}{\to}$' from here on — always give the same results.

Theorem 9.1 *We have $(Q \stackrel{C}{\to} R) = (Q \stackrel{HA}{\to} R)$, for any ZHA H and $Q, R \in H$.*

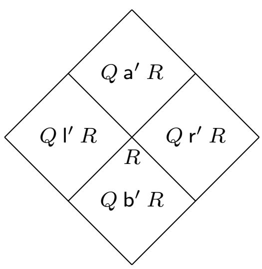
The proof will take the rest of this section, and our approach will be to check that for any ZHA H and $Q, R \in H$ this holds, for all $P \in H$:

$$(P \leq (Q \stackrel{\mathrm{C}}{\to} R)) \leftrightarrow (P \wedge Q \leq R).$$

In '$\stackrel{\mathrm{C}}{\to}$' the order of the cases is very important. For example, if $cd = 21$ and $ef = 23$ then both "$cd\,\mathsf{below}\,ef$" and "$cd\,\mathsf{leftof}\,ef$" are true, but "$cd\,\mathsf{below}\,ef$" takes precedence and so $cd \stackrel{\mathrm{C}}{\to} ef = \top$. We can fix this by creating variants of below, leftof, righof and above, called below', leftof', righof' and above', that make the four cases disjoint. Abbreviating below, leftof, righof and above as b, l, r and a, we have:

$$\begin{array}{rcl@{\qquad}rcl}
cd\ \mathsf{b}\ ef & := & c \leq e \wedge d \leq f & cd\ \mathsf{b}'\ ef & := & c \leq e \wedge d \leq f \\
cd\ \mathsf{l}\ ef & := & c \leq e \wedge d \geq f & cd\ \mathsf{l}'\ ef & := & c \leq e \wedge d > f \\
cd\ \mathsf{r}\ ef & := & c \geq e \wedge d \leq f & cd\ \mathsf{r}'\ ef & := & c > e \wedge d \leq f \\
cd\ \mathsf{a}\ ef & := & c > e \wedge d > f & cd\ \mathsf{a}'\ ef & := & c > e \wedge d > f
\end{array}$$

visually the regions are these, for R fixed:



Note that R belongs to the lower region — i.e., $R\,\mathsf{b}'\,R$.

[1] "When in doubt use brute force" — Ken Thompson

Now we clearly have:

$$Q \overset{\mathrm{C}}{\to} R \;=\; \left(\begin{array}{llll} \mathsf{if} & Q \,\mathsf{b}\, R & \mathsf{then} & \top \\ \mathsf{elseif} & Q \,\mathsf{l}\, R & \mathsf{then} & \mathsf{ne}(R) \\ \mathsf{elseif} & Q \,\mathsf{r}\, R & \mathsf{then} & \mathsf{nw}(R) \\ \mathsf{elseif} & Q \,\mathsf{a}\, R & \mathsf{then} & R \\ \mathsf{end} & & & \end{array} \right) = \left(\begin{array}{llll} \mathsf{if} & Q \,\mathsf{b}'\, R & \mathsf{then} & \top \\ \mathsf{elseif} & Q \,\mathsf{l}'\, R & \mathsf{then} & \mathsf{ne}(R) \\ \mathsf{elseif} & Q \,\mathsf{r}'\, R & \mathsf{then} & \mathsf{nw}(R) \\ \mathsf{elseif} & Q \,\mathsf{a}'\, R & \mathsf{then} & R \\ \mathsf{end} & & & \end{array} \right)$$

and $P \leq (Q \overset{\mathrm{C}}{\to} R)$ can be expressed as a conjunction of the four cases:

$$\begin{array}{ll} \multicolumn{2}{l}{((P \leq Q \overset{\mathrm{C}}{\to} R) \leftrightarrow (P \land Q \leq R))} \\ \leftrightarrow & \left(\begin{array}{ll} Q \,\mathsf{b}'\, R \to ((P \leq Q \overset{\mathrm{C}}{\to} R) \leftrightarrow (P \land Q \leq R)) & \land \\ Q \,\mathsf{l}'\, R \to ((P \leq Q \overset{\mathrm{C}}{\to} R) \leftrightarrow (P \land Q \leq R)) & \land \\ Q \,\mathsf{r}'\, R \to ((P \leq Q \overset{\mathrm{C}}{\to} R) \leftrightarrow (P \land Q \leq R)) & \land \\ Q \,\mathsf{a}'\, R \to ((P \leq Q \overset{\mathrm{C}}{\to} R) \leftrightarrow (P \land Q \leq R)) & \end{array} \right) \\ \\ \leftrightarrow & \left(\begin{array}{ll} Q \,\mathsf{b}'\, R \to ((P \leq \top) \leftrightarrow (P \land Q \leq R)) & \land \\ Q \,\mathsf{l}'\, R \to ((P \leq \mathsf{ne}(R)) \leftrightarrow (P \land Q \leq R)) & \land \\ Q \,\mathsf{r}'\, R \to ((P \leq \mathsf{nw}(R)) \leftrightarrow (P \land Q \leq R)) & \land \\ Q \,\mathsf{a}'\, R \to ((P \leq R) \leftrightarrow (P \land Q \leq R)) & \end{array} \right) \end{array}$$

Let's introduce a notation: a "$\widehat{a}$" means "make this digit as big possible without leaving the ZHA". So,

$$\text{in} \quad \begin{array}{cccccc} & 54 & & & & \\ 53 & & 44 & & & \\ & 43 & & 34 & & \\ 42 & & 33 & & 24 & \\ & 32 & & 23 & & \\ 31 & & 22 & & 13 & \\ & 21 & & 12 & & 03 \\ 20 & & 11 & & 02 & \\ & 10 & & 01 & & \\ & & 00 & & & \end{array} \quad \text{we have} \quad \begin{array}{rcccl} \widehat{1}\widehat{2} & = & 54 & = & \top, \\ 1\widehat{2} & = & 13 & = & \mathsf{ne}(12), \\ \widehat{1}2 & = & 42 & = & \mathsf{nw}(12); \end{array}$$

This lets us rewrite $\top$ as $\widehat{e}\widehat{f}$, $\mathsf{ne}(ef)$ as $e\widehat{f}$, and $\mathsf{nw}(ef)$ as $\widehat{e}f$.
Making $P = ab$, $Q = cd$, $R = ef$, we have:

$$
\begin{array}{l}
((ab \leq cd \stackrel{\mathrm{C}}{\to} ef) \leftrightarrow (ab \wedge cd \leq ef)) \\
\leftrightarrow \quad \left(\begin{array}{ll}
cd\,\mathsf{b}'\,ef \to ((ab \leq \widehat{\widehat{e}f}) \leftrightarrow (ab \wedge cd \leq ef)) & \wedge \\
cd\,\mathsf{l}'\,ef \to ((ab \leq e\widehat{f}) \leftrightarrow (ab \wedge cd \leq ef)) & \wedge \\
cd\,\mathsf{r}'\,ef \to ((ab \leq \widehat{e}f) \leftrightarrow (ab \wedge cd \leq ef)) & \wedge \\
cd\,\mathsf{a}'\,ef \to ((ab \leq ef) \leftrightarrow (ab \wedge cd \leq ef)) &
\end{array}\right) \\
\\
\leftrightarrow \quad \left(\begin{array}{ll}
c \leq e \wedge d \leq f \to ((ab \leq \widehat{\widehat{e}f}) \leftrightarrow (ab \wedge cd \leq ef)) & \wedge \\
c > e \wedge d \leq f \to ((ab \leq e\widehat{f}) \leftrightarrow (ab \wedge cd \leq ef)) & \wedge \\
c \leq e \wedge d > f \to ((ab \leq \widehat{e}f) \leftrightarrow (ab \wedge cd \leq ef)) & \wedge \\
c > e \wedge d > f \to ((ab \leq ef) \leftrightarrow (ab \wedge cd \leq ef)) &
\end{array}\right) \\
\\
\leftrightarrow \quad \left(\begin{array}{ll}
c \leq e \wedge d \leq f \to ((ab \leq \widehat{\widehat{e}f}) \leftrightarrow (ab \wedge cd \leq cd)) & \wedge \\
c > e \wedge d \leq f \to ((ab \leq e\widehat{f}) \leftrightarrow (ab \wedge cd \leq ed)) & \wedge \\
c \leq e \wedge d > f \to ((ab \leq \widehat{e}f) \leftrightarrow (ab \wedge cd \leq cf)) & \wedge \\
c > e \wedge d > f \to ((ab \leq ef) \leftrightarrow (ab \wedge cd \leq ef)) &
\end{array}\right) \\
\\
\leftrightarrow \quad \left(\begin{array}{ll}
c \leq e \wedge d \leq f \to ((ab \leq \widehat{\widehat{e}f}) \leftrightarrow \top) & \wedge \\
c > e \wedge d \leq f \to ((ab \leq e\widehat{f}) \leftrightarrow a \leq e) & \wedge \\
c \leq e \wedge d > f \to ((ab \leq \widehat{e}f) \leftrightarrow b \leq f) & \wedge \\
c > e \wedge d > f \to ((ab \leq ef) \leftrightarrow (a \leq e \wedge b \leq f)) &
\end{array}\right)
\end{array}
$$

In the last conjunction the four cases are trivial to check.

10 Logic in a Heyting Algebra

In sec.8 we saw a set of conditions — called 1 to 8' — that characterize the "Heyting-Algebra-ness" of a PC-structure. It is easy to see that Heyting-

Algebra-ness, or “HA-ness”, is equivalent to this set of conditions:

1)	$\forall P.$	$(P \le P)$					(id)
	$\forall P, Q, R.$	$(P \le R)$	$\leftarrow$	$(P \le Q)$	$\land$	$(Q \le R)$	(comp)
2)	$\forall P.$	$(P \le \top)$					$(\top_1)$
3)	$\forall Q.$	$(\bot \le Q)$					$(\bot_1)$
6)	$\forall P, Q, R.$	$(P \le Q \land R)$	$\rightarrow$	$(P \le Q)$			$(\land_1)$
	$\forall P, Q, R.$	$(P \le Q \land R)$	$\rightarrow$			$(P \le R)$	$(\land_2)$
	$\forall P, Q, R.$	$(P \le Q \land R)$	$\leftarrow$	$(P \le Q)$	$\land$	$(P \le R)$	$(\land_3)$
7)	$\forall P, Q, R.$	$(P \lor Q \le R)$	$\rightarrow$	$(P \le R)$			$(\lor_1)$
	$\forall P, Q, R.$	$(P \lor Q \le R)$	$\rightarrow$			$(Q \le R)$	$(\lor_2)$
	$\forall P, Q, R.$	$(P \lor Q \le R)$	$\leftarrow$	$(P \le R)$	$\land$	$(Q \le R)$	$(\lor_3)$
8)	$\forall P, Q, R.$	$(P \le Q \to R)$	$\rightarrow$	$(P \land Q \le R)$			$(\to_1)$
	$\forall P, Q, R.$	$(P \le Q \to R)$	$\leftarrow$	$(P \land Q \le R)$			$(\to_2)$

We omitted the conditions 4 and 5, that defined ‘$\leftrightarrow$’ and ‘$\neg$’ in terms of the other operators. The last column of the table gives a name to each of these new conditions.

These new conditions let us put (some) proofs about HAs in tree form, as we shall see soon.

Let us introduce two new notations. The first one,

$$(\text{expr}) \begin{bmatrix} v_1 {:=} \text{repl}_1 \\ v_2 {:=} \text{repl}_2 \end{bmatrix}$$

indicates simultaneous substitution of all (free) occurrences of the variables v_1 and v_2 in expr by the replacements repl_1 and repl_2. For example,

$$((x+y)\cdot z) \begin{bmatrix} x{:=}a{+}y \\ y{:=}b{+}z \\ z{:=}c{+}x \end{bmatrix} \quad = \quad ((a+y)+(b+z))\cdot(c+x).$$

The second is a way to write ‘$\to$’s as horizontal bars. In

$$\frac{A \quad B \quad C}{D}\alpha \qquad \frac{E \quad F}{G}\beta \qquad \frac{H}{I}\gamma \qquad \frac{}{J}\delta \qquad \frac{\dfrac{}{K}\epsilon \quad \dfrac{L \quad M}{N}\zeta \quad O}{P}\eta$$

the trees mean:

- if A, B, C are true then D is true (by α),

- if E, F, are true then G is true (by β),
- if H is true then I is true (by γ),
- J is true (by δ, with no hypotheses),
- K is true (by ϵ); if L and M then N (by ζ); if K, N, O, then P (by η); combining all this we get a way to prove that if L, M, O, then P,

where $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta$ are usually names of rules.

The implications in the table in the beginning of this section can be rewritten as "tree rules" as:

$$\frac{}{P \leq P}\,\mathrm{id} \qquad \frac{P \leq Q \quad Q \leq R}{P \leq R}\,\mathrm{comp} \qquad \frac{}{P \leq \top}\,\top_1 \qquad \frac{}{\bot \leq Q}\,\bot_1$$

$$\frac{P \leq Q \land R}{P \leq Q}\,\land_1 \qquad \frac{P \leq Q \land R}{P \leq R}\,\land_2 \qquad \frac{P \leq Q \quad P \leq R}{P \leq Q \land R}\,\land_3$$

$$\frac{P \lor Q \leq R}{P \leq R}\,\lor_1 \qquad \frac{P \lor Q \leq R}{Q \leq R}\,\lor_2 \qquad \frac{P \leq R \quad Q \leq R}{P \lor Q \leq R}\,\lor_3$$

$$\frac{P \leq Q \to R}{P \land Q \leq R}\,\to_1 \qquad \frac{P \land Q \leq R}{P \leq Q \to R}\,\to_2$$

Note that the '$\forall P, Q, R \in \Omega$'s are left implicit in the tree rules, which means that *every substitution instance of the tree rules hold*; sometimes — but rarely — we will indicate the substitution explicitly, like this,

$$\left(\frac{P \land Q \leq R}{P \leq Q \to R}\,\to_2\right)\left[\begin{smallmatrix} Q:=P\to\bot \\ R:=\bot \end{smallmatrix}\right] \quad \rightsquigarrow \quad \frac{P \land (P\to\bot) \leq \bot}{P \leq ((P\to\bot)\to\bot)}\,\to_2$$

$$(\to_2)\left[\begin{smallmatrix} Q:=P\to\bot \\ R:=\bot \end{smallmatrix}\right] \quad \rightsquigarrow \quad \frac{P \land (P\to\bot) \leq \bot}{P \leq ((P\to\bot)\to\bot)}\,\to_2\left[\begin{smallmatrix} Q:=P\to\bot \\ R:=\bot \end{smallmatrix}\right]$$

Usually we will only say '$\to_2$' instead of '$\to_2 \left[\begin{smallmatrix} Q:=P\to\bot \\ R:=\bot \end{smallmatrix}\right]$' at the right of a bar, and the task of discovering which substitution has been used is left to the reader.

The tree rules can be composed in a nice visual way. For example, this tree — let's call it $(\land\land)$,

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P\land Q \le P\land Q}\,\mathrm{id}}{P\land Q\le P}\,\land_1 \qquad P\le R}{P\land Q\le R}\,\mathrm{comp} \qquad \dfrac{\dfrac{\dfrac{}{P\land Q\le P\land Q}\,\mathrm{id}}{P\land Q\le Q}\,\land_2 \qquad Q\le S}{P\land Q\le S}\,\mathrm{comp}}{P\land Q\le R\land S}\,\land_3$$

"is" a proof for:

$$\forall P, Q, R, S \in \Omega.\ (P \le R) \land (Q \le S) \to ((P \land Q) \le (R \land S)).$$

We can perform substitutions on trees, and the notation will be the same as for tree rules: for example, $(\land\land)\,[\,{\scriptstyle S:=P\land Q}\,]$.

10.1 Derived rules

Let be HAT the set of "Heyting Algebra rules in tree form" from the last section:

$$\mathsf{HAT} = \{(\mathrm{id}), \ldots, (\to_2)\}.$$

Let's see a way to treat HAT as a deductive system.
If S is a set of tree rules, we will write:

- $\mathsf{Trees}(\mathsf{S})$ for the set of all trees whose bars are all substituion instances of rules in S,
- $\mathsf{Trees}(\mathsf{S}, \{H_1, \ldots, H_n\})$ for the set of all trees in $\mathsf{Trees}(\mathsf{S})$ whose hypotheses are contained in the set $\{H_1, \ldots, H_n\}$, and
- $\mathsf{Trees}(\mathsf{S}, \{H_1, \ldots, H_n\}, C)$ for the set of trees in $\mathsf{Trees}(\mathsf{S}, \{H_1, \ldots, H_n\})$ having C as their conclusion.

When the set S is clear from the context, we write

$$\begin{array}{c} H_1 \quad \ldots \quad H_n \\ \hline\hline C \end{array}$$

to mean: *we know* a tree in $\mathsf{Trees}(\mathsf{S}, \{H_1, \ldots, H_n\}, C)$, and this is an abbreviation for it. I like to think of the double bar as the bellows of a closed accordion: when the accordion is closed we can still see the keyboards at both sides, but not the drawings painted on the folded part of the pleated cloth.

The notation that defines a *derived rule* is "*newrule* := *expansion*", where *expansion* is a tree in $\mathsf{Trees}(\mathsf{S}, \{H_1, \ldots, H_n\}, C)$ and *newrule* is a bar with hypotheses $H_1, \ldots, H_n$ and conclusion C, written with a single bar with a (new) rule name, instead of with a double bar. For example, this is a version of Modus Ponens for Heyting Algebras:

$$\dfrac{P \le Q \quad P \le Q{\to}R}{P \le R}\,\mathsf{MP} \qquad := \qquad \dfrac{\dfrac{P \le Q{\to}R \quad P \le Q}{P \le (Q{\to}R){\wedge}Q}\,\wedge_3 \quad \dfrac{\dfrac{}{Q{\to}R \le Q{\to}R}\,\mathrm{id}}{(Q{\to}R){\wedge}Q \le R}\,{\to}_1}{P \le R}\,\mathrm{comp}$$

After the definition of a derived rule — say, "$D_1 := E_1$" — the set of allowed tree rules that is implicit from the context is increased, with D_1 being added to it; when we define another derived rule, $D_2 := E_2$, its expansion E_2 can have bars that are substitution instances of D_1. After adding more derived rules, $D_3 := E_3$, …, $D_n := E_n$, we can use all the new rules $D_1, \ldots, D_n$ in our trees — and we have a way to remove all the derived rules from our trees! Take a tree $T \in \mathsf{Trees}(\mathsf{S} \cup \{D_1, \ldots, D_n\})$; replace each substitution instance of D_n in it by its expansion, then replace every substitution instance of D_{n-1} in the new tree by its expansion, and so on; after replacing all substitution instances of D_1 we get a tree in $\mathsf{Trees}(\mathsf{S})$, with the same hypotheses and the same conclusion as the original T.

We want to add these other derived rules:

$$\dfrac{}{Q \wedge R \le Q}\,{\wedge}E_1 \qquad := \qquad \dfrac{\dfrac{}{Q \wedge R \le Q \wedge R}\,\mathrm{id}}{Q \wedge R \le Q}\,\wedge_1$$

$$\dfrac{}{Q \wedge R \le R}\,{\wedge}E_2 \qquad := \qquad \dfrac{\dfrac{}{Q \wedge R \le Q \wedge R}\,\mathrm{id}}{Q \wedge R \le R}\,\wedge_2$$

$$\dfrac{}{P \le P \vee Q}\,{\vee}I_1 \qquad := \qquad \dfrac{\dfrac{}{P \vee Q \le P \vee Q}\,\mathrm{id}}{P \le P \vee Q}\,\vee_1$$

$$\dfrac{}{Q \le P \vee Q}\,{\vee}I_2 \qquad := \qquad \dfrac{\dfrac{}{P \vee Q \le P \vee Q}\,\mathrm{id}}{Q \le P \vee Q}\,\vee_2$$

$$\dfrac{P \wedge R \le S \quad Q \wedge R \le S}{(P \vee Q) \wedge R \le R}\,{\vee}E \qquad := \qquad \dfrac{\dfrac{\dfrac{P \wedge R \le S}{P \le R \to S}\,{\to}_2 \quad \dfrac{Q \wedge R \le S}{Q \le R \to S}\,{\to}_2}{P \vee Q \le R \to S}\,\vee_3}{(P \vee Q) \wedge R \le R}\,{\to}_1$$

10.2 Natural deduction

The system HAT with all the derived rules of the last section added to it will be called HAND:

$$\mathsf{HAND} = \{(\mathrm{id}), \ldots, (\to_2), (\mathsf{MP}), \ldots, (\vee E)\}$$

Trees in Natural Deduction for IPL can be translated into HAND by a method that we will sketch below. Note that this section is not self-contained — it should be regarded as an introduction to [NP01]. Note that *all our trees can be intepreted as proofs about Heyting Algebras.*

This is an example of a tree in Natural Deduction:

$$\dfrac{\dfrac{[P]^1 \quad P{\to}Q}{Q}\;(\to E) \qquad \dfrac{[P]^1 \quad P{\to}R}{R}\;(\to E)}{\dfrac{Q{\wedge}R}{P{\to}(Q{\wedge}R)}\;(\to I);1}\;(\wedge I)$$

The ";1" in its last bar means: below this point the hypotheses marked with '$[\,\cdot\,]^1$' are "discharged" from the list of hypotheses. Each subtree of a ND tree with undischarged hypotheses $H_1, \ldots, H_n$ and conclusion C will be interpreted as *some* tree in HAND with no hypotheses and conclusion $H_1 \wedge \ldots \wedge H_n \leq C$ — there are usually several possible choices. So:

$$\dfrac{P \quad P{\to}Q}{Q} \quad \Longrightarrow \quad \dfrac{}{P \wedge (P{\to}Q) \leq Q}\;\mathsf{MP}$$

$$\dfrac{P \quad P{\to}R}{R} \quad \Longrightarrow \quad \dfrac{}{P \wedge (P{\to}R) \leq R}\;\mathsf{MP}$$

$$\dfrac{Q \quad R}{Q{\wedge}R} \quad \Longrightarrow \quad \dfrac{}{Q{\wedge}R \leq Q{\wedge}R}\;\mathrm{id}$$

$$\dfrac{\dfrac{P \quad P{\to}Q}{Q} \qquad \dfrac{P \quad P{\to}R}{R}}{Q{\wedge}R} \quad \Longrightarrow \quad \overline{\overline{((P{\to}R) \wedge (P{\to}Q)) \wedge P \leq Q{\wedge}R}}$$

$$\dfrac{\dfrac{\dfrac{[P]^1 \quad P{\to}Q}{Q} \qquad \dfrac{[P]^1 \quad P{\to}R}{R}}{Q{\wedge}R}}{P{\to}(Q{\wedge}R)}\;(\to I);1 \quad \Longrightarrow \quad \dfrac{\overline{\overline{((P{\to}R) \wedge (P{\to}Q)) \wedge P \leq Q{\wedge}R}}}{(P{\to}R) \wedge (P{\to}Q) \leq P{\to}Q{\wedge}R}\;{\to_2}$$

The ND rules that are difficult to understand and difficult to translate are the ones that involve discharges: '$(\to I)$', that appears above, and '$(\lor E)$':

$$\dfrac{P\lor Q \qquad \begin{matrix}[P]^1 \quad R\\ \vdots\, T_1\\ S\end{matrix} \qquad \begin{matrix}[Q]^1 \quad R\\ \vdots\, T_2\\ S\end{matrix}}{S}\,(\lor E) \quad\Longrightarrow\quad \dfrac{\overline{\overline{P\land R\le S}}\,T_1 \qquad \overline{\overline{Q\land R\le S}}\,T_2}{(P\lor Q)\land R\le S}\,\lor E$$

Note that the derived rule $\lor E$ is used to combine the translations of the subtrees T_1 and T_2 into a translation of the whole ND tree.

My suggestion for the readers that are seeing this for the first time is: start by translating the ND tree below

$$\dfrac{\dfrac{(P\lor Q)\land R}{P\lor Q}\,(\land E_1) \qquad \dfrac{\dfrac{[P]^1 \quad \dfrac{(P\lor Q)\land R}{R}\,(\land E_2)}{P\land R}\,(\land I)}{(P\land R)\lor(Q\land R)}\,(\lor I_1) \qquad \dfrac{\dfrac{[Q]^1 \quad \dfrac{(P\lor Q)\land R}{R}\,(\land E_2)}{Q\land R}\,(\land I)}{(P\land R)\lor(Q\land R)}\,(\lor I_2)}{(P\land R)\lor(Q\land R)}\,(\lor E);1$$

to a tree in HAND, and then to a tree in HAT; then read the relevant parts of [NP01] to see how they would do that translation.

11 Topologies

The best way to connect ZHAs to several standard concepts is by seeing that ZHAs are topologies on certain finite sets — actually on 2-column acyclical graphs (sec.14). This will be done here and in the next few sections.

A *topology* on a set X is a subset $\mathcal{U}$ of $\mathcal{P}(X)$ that contains the "everything" and the "nothing" and is closed by binary unions and intersections and by arbitrary unions. Formally:

1) $\mathcal{U}$ contains X and $\varnothing$,

2) if $P, Q \in \mathcal{U}$ then $\mathcal{U}$ contains $P \cup Q$ and $P \cap Q$,

3) if $\mathcal{V} \subset \mathcal{U}$ then $\mathcal{U}$ contains $\bigcup \mathcal{V}$.

A *topological space* is a pair $(X, \mathcal{U})$ where X is a set and $\mathcal{U}$ is a topology on X.

When $(X, \mathcal{U})$ is a topological space and $U \in \mathcal{U}$ we say that U is *open* in $(X, \mathcal{U})$.

For example, let X be the ZSet $\begin{smallmatrix}\bullet&&\bullet\\&\bullet&\\\bullet&&\bullet\end{smallmatrix}$, and let's use the characteristic function notation from sec.1 to denote its subsets — we write $X = \begin{smallmatrix}1&&1\\&1&\\1&&1\end{smallmatrix}$ and $\varnothing = \begin{smallmatrix}0&&0\\&0&\\0&&0\end{smallmatrix}$ instead of $X = \begin{smallmatrix}\bullet&&\bullet\\&\bullet&\\\bullet&&\bullet\end{smallmatrix}$ and $\varnothing = \begin{smallmatrix}\cdot&&\cdot\\&\cdot&\\\cdot&&\cdot\end{smallmatrix}$.

If $\mathcal{U} = \left\{ \begin{smallmatrix}1&&0\\&0&\\0&&0\end{smallmatrix}, \begin{smallmatrix}0&&1\\&0&\\0&&0\end{smallmatrix}, \begin{smallmatrix}0&&0\\&1&\\0&&0\end{smallmatrix}, \begin{smallmatrix}0&&0\\&0&\\1&&0\end{smallmatrix}, \begin{smallmatrix}0&&0\\&0&\\0&&1\end{smallmatrix} \right\}$ then $\mathcal{U} \subset \mathcal{P}(X)$ but $\mathcal{U}$ fails all the conditions in 1, 2, 3 above:

1) $X = \begin{smallmatrix}1&&1\\&1&\\1&&1\end{smallmatrix} \notin \mathcal{U}$ and $\varnothing = \begin{smallmatrix}0&&0\\&0&\\0&&0\end{smallmatrix} \notin \mathcal{U}$

2) Let $P = \begin{smallmatrix}1&&0\\&0&\\0&&0\end{smallmatrix} \in \mathcal{U}$ and $Q = \begin{smallmatrix}0&&1\\&0&\\0&&0\end{smallmatrix} \in \mathcal{U}$. Then $P \cap Q = \begin{smallmatrix}0&&0\\&0&\\0&&0\end{smallmatrix} \notin \mathcal{U}$ and $P \cup Q = \begin{smallmatrix}1&&1\\&0&\\0&&0\end{smallmatrix} \notin \mathcal{U}$.

3) Let $\mathcal{V} = \left\{ \begin{smallmatrix}0&&1\\&0&\\0&&0\end{smallmatrix}, \begin{smallmatrix}0&&0\\&1&\\0&&0\end{smallmatrix}, \begin{smallmatrix}0&&0\\&0&\\1&&0\end{smallmatrix} \right\} \subset \mathcal{U}$. Then $\bigcup \mathcal{V} = \begin{smallmatrix}0&&1\\&0&\\0&&0\end{smallmatrix} \cup \begin{smallmatrix}0&&0\\&1&\\0&&0\end{smallmatrix} \cup \begin{smallmatrix}0&&0\\&0&\\1&&0\end{smallmatrix} = \begin{smallmatrix}0&&1\\&1&\\1&&0\end{smallmatrix} \notin \mathcal{U}$.

Now let $K = \begin{smallmatrix}&\bullet&\\\bullet&&\bullet\\&\bullet&\\&\bullet&\end{smallmatrix}$ and $\mathcal{U} = \left\{ \begin{smallmatrix}&0&\\0&&0\\&0&\\&0&\end{smallmatrix}, \begin{smallmatrix}&0&\\0&&0\\&0&\\&1&\end{smallmatrix}, \begin{smallmatrix}&0&\\0&&0\\&1&\\&1&\end{smallmatrix}, \begin{smallmatrix}&0&\\0&&1\\&1&\\&1&\end{smallmatrix}, \begin{smallmatrix}&0&\\1&&0\\&1&\\&1&\end{smallmatrix}, \begin{smallmatrix}&0&\\1&&1\\&1&\\&1&\end{smallmatrix}, \begin{smallmatrix}&1&\\1&&1\\&1&\\&1&\end{smallmatrix} \right\}$. In this case $(K,\mathcal{U})$ is a topological space.

Some sets have "default" topologies on them, denoted with '$\mathcal{O}$'. For example, $\mathbb{R}$ is often used to mean the topological space $(\mathbb{R}, \mathcal{O}(\mathbb{R}))$, where:

$$\mathcal{O}(\mathbb{R}) = \{\, U \subset \mathbb{R} \mid U \text{ is a union of open intervals} \,\}.$$

We say that a subset $U \subset \mathbb{R}$ is "open in $\mathbb{R}$" ("in the default sense"; note that now we are saying just "open in $\mathbb{R}$", not "open in $(\mathbb{R}, \mathcal{O}(\mathbb{R}))$") when U is a union of open intervals, i.e., when $U \in \mathcal{O}(\mathbb{R})$; but note that $\mathcal{P}(\mathbb{R})$ and $\{\varnothing, \mathbb{R}\}$ are also topologies on $\mathbb{R}$, and:

$$\begin{array}{lll}
\{2,3,4\} \in \mathcal{P}(\mathbb{R}), & \text{so} & \{2,3,4\} \text{ is open in } (\mathbb{R}, \mathcal{P}(\mathbb{R})), \\
\{2,3,4\} \notin \mathcal{O}(\mathbb{R}), & \text{so} & \{2,3,4\} \text{ is not open in } (\mathbb{R}, \mathcal{O}(\mathbb{R})), \\
\{2,3,4\} \notin \{\varnothing, \mathbb{R}\}, & \text{so} & \{2,3,4\} \text{ is not open in } (\mathbb{R}, \{\varnothing, \mathbb{R}\});
\end{array}$$

when we say just "U is open in X", this means that:

1) $\mathcal{O}(X)$ is clear from the context, and

2) $U \in \mathcal{O}(X)$.

12 The default topology on a ZSet

Let's define a default topology $\mathcal{O}(D)$ for each ZSet D.

For each ZSet D we define $\mathcal{O}(D)$ as:

$$\begin{array}{rcl}
\mathcal{O}(D) & := & \{\, U \subset D \mid \forall((x,y),(x',y')) \in \mathsf{BPM}(D). \\
& & \qquad\qquad (x,y) \in U \to (x',y') \in U \,\}
\end{array}$$

whose visual meaning is this. Turn D into a ZDAG by adding arrows for the black pawns moves (sec.2), and regard each subset $U \subset D$ as a board configuration in which the black pieces may move down to empty positions through the arrows. A subset U is "stable" when no moves are possible because all points of U "ahead" of a black piece are already occupied by black pieces;

a subset U is "non-stable" when there is at least one arrow $((x,y),(x',y')) \in \mathsf{BPM}(D)$ in which (x,y) had a black piece and (x',y') is an empty position.

In our two notations for subsets (sec.1) a subset $U \subset D$ is unstable when it has an arrow like '$\bullet \to \cdot$' or '$1 \to 0$'; remember that black pawn moves arrows go down. A subset $U \subset D$ is stable when none of its '$\bullet$'s or '1's can move down to empty positions.

"Open" is the same as "stable". $\mathcal{O}(D)$ is the set of stable subsets of D.

Some examples:
$\begin{smallmatrix}&0&\\0&&1\\&0&\\&0&\end{smallmatrix}$ is not open because it has a 1 above a 0,
$\mathcal{O}(\begin{smallmatrix}&\bullet&\\\bullet&&\bullet\\&\bullet&\\&\bullet&\end{smallmatrix}) = \left\{ \begin{smallmatrix}&0&\\0&&0\\&0&\\&0&\end{smallmatrix}, \begin{smallmatrix}&0&\\0&&0\\&0&\\&1&\end{smallmatrix}, \begin{smallmatrix}&0&\\0&&0\\&1&\\&1&\end{smallmatrix}, \begin{smallmatrix}&0&\\0&&1\\&1&\\&1&\end{smallmatrix}, \begin{smallmatrix}&0&\\1&&0\\&1&\\&1&\end{smallmatrix}, \begin{smallmatrix}&0&\\1&&1\\&1&\\&1&\end{smallmatrix}, \begin{smallmatrix}&1&\\1&&1\\&1&\\&1&\end{smallmatrix} \right\}$,
$\mathcal{O}(\begin{smallmatrix}&\bullet&\\\bullet&&\bullet\\\bullet&&\bullet\end{smallmatrix}) = \left\{ \begin{smallmatrix}&0&\\0&&0\\0&&0\end{smallmatrix}, \begin{smallmatrix}&0&\\0&&0\\0&&1\end{smallmatrix}, \begin{smallmatrix}&0&\\0&&0\\1&&0\end{smallmatrix}, \begin{smallmatrix}&0&\\0&&0\\1&&1\end{smallmatrix}, \begin{smallmatrix}&0&\\0&&1\\0&&1\end{smallmatrix}, \begin{smallmatrix}&0&\\0&&1\\1&&1\end{smallmatrix}, \begin{smallmatrix}&0&\\1&&0\\1&&0\end{smallmatrix}, \begin{smallmatrix}&0&\\1&&0\\1&&1\end{smallmatrix}, \begin{smallmatrix}&0&\\1&&1\\1&&1\end{smallmatrix}, \begin{smallmatrix}&1&\\1&&1\\1&&1\end{smallmatrix} \right\}$.

The definition of $\mathcal{O}(D)$ above can be generalized to any directed graph. If (A,R) is a directed graph, then $(A,\mathcal{O}_R(A))$ is a topological space if we define:

$$\mathcal{O}_R(A) := \{\, U \subseteq A \mid \forall (a,b) \in R.\ (a \in U \to b \in U) \,\}$$

The two definitions are related as this: $\mathcal{O}(D) = \mathcal{O}_{\mathsf{BPM}(D)}(D)$.

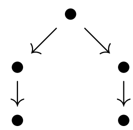
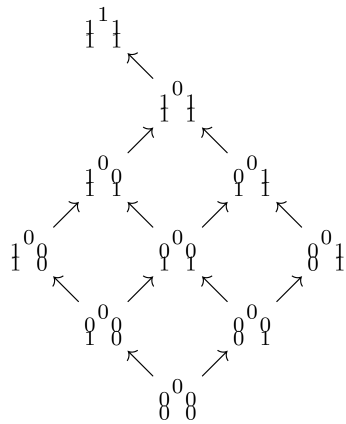
Note that we can see the arrows in $\mathsf{BPM}(D)$ or in R as *obligations* that open sets must obey; each arrow $a \to b$ says that every open set that contains a is forced to contain b too.

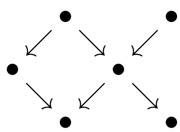
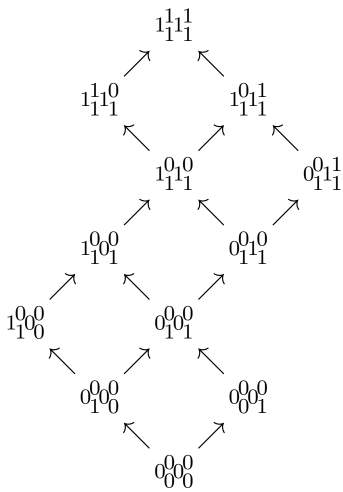
13 Topologies as partial orders

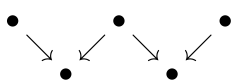
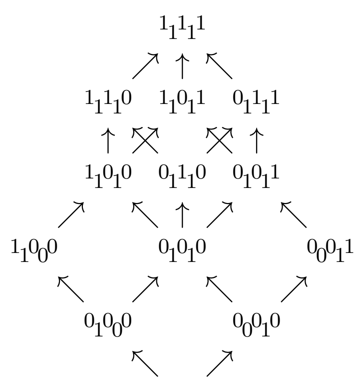
For any topological space $(X,\mathcal{O}(X))$ we can regard $\mathcal{O}(X)$ as a partial order, ordered by inclusion, with $\varnothing$ as its minimal element and X as its maximal element; we denote that partial order by $(\mathcal{O}(X),\subseteq)$.

Take any ZSet D. The partial order $(\mathcal{O}(D),\subseteq)$ will *sometimes* be a ZHA when we draw it with $\varnothing$ at the bottom, D at the top, and inclusions pointing up, as can be seen in the three figures below; when $D = \begin{smallmatrix}&\bullet&\\\bullet&&\bullet\\\bullet&&\bullet\end{smallmatrix}$ or $D = \begin{smallmatrix}&\bullet&&\bullet\\\bullet&&\bullet&\\&\bullet&&\bullet\end{smallmatrix}$ the result is a ZHA, but when $D = \begin{smallmatrix}\bullet&&\bullet&&\bullet\\&\bullet&&\bullet&\end{smallmatrix}$ it is not.

Let's write "$V \subset_1 U$" for "$V \subseteq U$ and V and U differ in exactly one point". When D is a ZSet the relation $\subseteq$ on $\mathcal{O}(D)$ is the transitive-reflexive closure of $\subset_1$, and $(\mathcal{O}(D),\subset_1)$ is easier to draw than $(\mathcal{O}(D),\subseteq)$.

$(H, \mathsf{BPM}(H)) =$  $(\mathcal{O}(H), \subset_1) =$ 

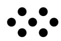
$(G, \mathsf{BPM}(G)) =$  $(\mathcal{O}(G), \subset_1) =$ 

$(W, \mathsf{BPM}(W)) =$  $(\mathcal{O}(W), \subset_1) =$ 

We can formalize a "way to draw $\mathcal{O}(D)$ as a ZHA" (or "...as a ZDAG") as a bijective function f from a ZHA (or from a ZSet) S to $\mathcal{O}(D)$ that creates a perfect correspondence between the white moves in S and the "$V \subset_1 U$-arrows"; more precisely, an f such that this holds: if $a, b \in S$ then $(a, b) \in \mathsf{WPM}(S)$ iff $f(a) \subset_1 f(b)$.

Note that the *number of elements* in an open set corresponds to the *height* where it is drawn; if $f : S \to \mathcal{O}(D)$ is a way to draw $\mathcal{O}(D)$ as a ZHA or a ZDAG then f takes points of the form $(\_\_, y)$ to open sets with y elements, and if $f : S \to \mathcal{O}(D)$ is a way to draw $\mathcal{O}(D)$ as a ZHA (not a ZDAG!) then we also have that $f((0, 0)) = \varnothing \in \mathcal{O}(D)$.

The diagram for $(\mathcal{O}(H), \subset_1)$ above is a way to draw $\mathcal{O}(H)$ as a ZHA.
The diagram for $(\mathcal{O}(G), \subset_1)$ above is a way to draw $\mathcal{O}(G)$ as a ZHA.
The diagram for $(\mathcal{O}(W), \subset_1)$ above is *not* a way to draw $\mathcal{O}(W)$ as a ZSet. Look at ${}^0{}_1{}^1{}_1{}^0$ and ${}^1{}_1{}^0{}_1{}^1$ in the middle of the cube formed by all open sets of the form ${}^a{}_1{}^b{}_1{}^c$. We don't have ${}^0{}_1{}^1{}_1{}^0 \subset_1 {}^1{}_1{}^0{}_1{}^1$, but we do have a white pawn move (not draw in the diagram!) from $f^{-1}({}^0{}_1{}^1{}_1{}^0)$ to $f^{-1}({}^1{}_1{}^0{}_1{}^1)$. We say that a ZSet is *thin* when it doesn't have three independent points.

Every time that a ZSet D has three independent points, as in W, we will have a situation like in $(\mathcal{O}(W), \subset_1)$; for example, if $B =$  then the open sets of B of the form $a{}^0_1b{}^0_1c$ form a cube.

14 2-Column Graphs

Note: in this section we will manipulate objects with names like $1\_, 2\_, 3\_, \ldots,$ $\_1, \_2, \_3, \ldots$; here are two good ways to formalize them:

$$\begin{array}{ll} \vdots & \vdots \\ 4\_ = (0,4) & \_4 = (1,4) \\ 3\_ = (0,3) & \_3 = (1,3) \\ 2\_ = (0,2) & \_2 = (1,2) \\ 1\_ = (0,1) & \_1 = (1,1) \end{array} \quad \text{or} \quad \begin{array}{ll} \vdots & \vdots \\ 4\_ = \texttt{"4\_"} & \_4 = \texttt{"\_4"} \\ 3\_ = \texttt{"3\_"} & \_3 = \texttt{"\_3"} \\ 2\_ = \texttt{"2\_"} & \_2 = \texttt{"\_2"} \\ 1\_ = \texttt{"1\_"} & \_1 = \texttt{"\_1"} \end{array},$$

where `"1_"`, `"_2"`, `""`, `"Hello!"`, etc are strings.

We define:

$$\begin{array}{rcl} \mathsf{LC}(l) & := & \{1\_, 2\_, \ldots, l\_\} \\ \mathsf{RC}(r) & := & \{\_1, \_2, \ldots, \_r\}, \end{array}$$

which generate a "left column" of height l and a "right column" of height r.

A *description for a 2-column graph* (a "D2CG") is a 4-tuple (l, r, R, L), where $l, r \in \mathbb{N}$, $R \subset \mathsf{LC}(l) \times \mathsf{RC}(r)$, $L \subset \mathsf{RC}(r) \times \mathsf{LC}(l)$; l is the height of the left column, r is the height of the right column, and R and L are set of intercolumn arrows (going right and left respectively).

The operation $\mathsf{2CG}$ (in a sans-serif font) generates a directed graph from a D2CG:

$$\mathsf{2CG}(l, r, R, L) \quad := \quad \left(\mathsf{LC}(l) \cup \mathsf{RC}(r), \left\{ \begin{array}{c} \{l\_{\to}(l{-}1)\_, \ ..., \ 2\_{\to}1\_\}\cup \\ \{\_r{\to}\_(r{-}1), \ ..., \ \_2{\to}\_1\}\cup \\ R{\cup}L \end{array} \right\} \right)$$

For example,

$$\mathsf{2CG}(3, 4, \left\{ \begin{array}{c} 3\_{\to}\_4, \\ 2\_{\to}\_3 \end{array} \right\}, \left\{ \begin{array}{c} 2\_{\leftarrow}\_2, \\ 1\_{\leftarrow}\_2 \end{array} \right\}) \quad := \quad \left(\left\{ \begin{array}{r} 3\_, \ 2\_, \ 1\_, \\ \_4, \ \_3, \ \_2, \ \_1 \end{array} \right\}, \left\{ \begin{array}{c} 3\_{\to}2\_, \ 2\_{\to}1\_, \\ \_4{\to}\_3, \ \_3{\to}\_2, \ \_2{\to}\_1, \\ 3\_{\to}\_4, \ 2\_{\to}\_3, \\ 2\_{\leftarrow}\_2, \ 1\_{\leftarrow}\_2 \end{array} \right\} \right)$$

which is:

$$\left(\begin{tikzcd} & \_4 \arrow[d] \\ 3\_ \arrow[ur] \arrow[d] & \_3 \arrow[d] \\ 2\_ \arrow[ur] \arrow[d] & \_2 \arrow[l] \arrow[dl] \arrow[d] \\ 1\_ & \_1 \end{tikzcd} \right)$$

we will usually draw that more compactly, by omitting the intracolumn (i.e., vertical) arrows:

$$\left(\begin{tikzcd}[sep=tiny] & \_4 \\ 3\_ \arrow[ur] & \_3 \\ 2\_ \arrow[ur] & \_2 \arrow[l] \arrow[dl] \\ 1\_ & \_1 \end{tikzcd} \right) \quad \text{or} \quad \left(\begin{tikzcd}[sep=tiny] & \bullet \\ \bullet \arrow[ur] & \bullet \\ \bullet \arrow[ur] & \bullet \arrow[l] \arrow[dl] \\ \bullet & \bullet \end{tikzcd} \right).$$

A *2-column graph* (a "2CG") is a directed graph that is of the form $\mathsf{2CG}(l, r, R, L)$. We will often say $(P, A) = \mathsf{2CG}(l, r, R, L)$, where the P stand for "points" and A for "arrows".

A *2-column acyclical graph* (a "2CAG") is a 2CG that doesn't have cycles. If L has an arrow that is the opposite of an arrow in R, this generates a cycle of length 2; if R has an arrow $l\_{\to}\_r'$ and L has an arrow $l'\_{\leftarrow}\_r$, where $l \le l'$ and $r \le r'$, this generates a cycle that can have a more complex shape — a triangle or a bowtie. For example,

$$\left(\begin{tikzcd} 4\_ \arrow[d] & \\ 3\_ \arrow[d] \arrow[dr, leftrightarrow] & \_3 \arrow[d] \\ 2\_ \arrow[d] & \_2 \arrow[d] \\ 1\_ & \_1 \end{tikzcd} \right) \quad \text{and} \quad \left(\begin{tikzcd} & \_4 \arrow[d] \\ 3\_ \arrow[d] & \_3 \arrow[d] \\ 2\_ \arrow[d] \arrow[ur] & \_2 \arrow[d] \arrow[ul] \\ 1\_ & \_1 \end{tikzcd} \right).$$

15 Topologies on 2CGs

In this section we will see that ZHAs are topologies on 2CAGs.

Let $(P, A) = \mathsf{2CG}(l, r, R, L)$ be a 2-column graph.
What happens if we look at the open sets of (P, A), i.e., at $\mathcal{O}_A(P)$? Two things:
1) every open set $U \in \mathcal{O}_A(P)$ is of the form $\mathsf{LC}(a) \cup \mathsf{RC}(b)$,
2) arrows in R and L forbids some '$\mathsf{LC}(a) \cup \mathsf{RC}(b)$'s from being open sets.
In order to understand that we need to introduce some notations for "piles".

The function

$$\mathsf{pile}(\langle a, b\rangle) := \mathsf{LC}(a) \cup \mathsf{RC}(b)$$

converts an element $\langle a, b\rangle \in \mathbb{LR}$ into a pile of elements in the left column of height a and a pile of elements in the right column of height b. We will write subsets of the points of a 2CG using a positional notation with arrows. So, for example, if $(P, A) = \mathsf{2CG}(3, 4, \{2\_{\rightarrow}\_3\}, \{2\_{\leftarrow}\_2\})$ then

$$(P, A) = \left(\begin{array}{ccc} & & \_4 \\ 3\_ & \nearrow & \_3 \\ 2\_ & \leftarrow & \_2 \\ 1\_ & & \_1 \end{array}\right) \qquad \text{and} \qquad \mathsf{pile}(21) = \left(\begin{array}{ccc} & & 0 \\ 0 & \nearrow & 0 \\ 1 & \leftarrow & 0 \\ 1 & & 1 \end{array}\right) \quad \text{(as a subset of } P\text{).}$$

Note that $\mathsf{pile}(21)$ is not open in $(P, \mathcal{O}_A(P))$, as it has an arrow '$1 \to 0$'. In fact, the presence of the arrow $\{2\_{\rightarrow}\_3\}$ in A means that all piles of the form

$$\left(\begin{array}{ccc} & & 0 \\ ? & \nearrow & 0 \\ 1 & \leftarrow & ? \\ 1 & & ? \end{array}\right)$$

are not open, the presence of the arrow $\{2\_{\leftarrow}\_2\}$ means that the piles of the form

$$\left(\begin{array}{ccc} & & ? \\ 0 & \nearrow & ? \\ 0 & \leftarrow & 1 \\ ? & & 1 \end{array}\right)$$

are not open sets.

The effect of these prohibitions can be expressed nicely with implications. If

$$(P, A) = \mathsf{2CG}(l, r, \left\{\begin{array}{c} c\_{\rightarrow}\_d, \\ e\_{\rightarrow}\_f \end{array}\right\}, \left\{\begin{array}{c} g\_{\leftarrow}\_h, \\ i\_{\leftarrow}\_j \end{array}\right\})$$

then

$$\mathcal{O}_A(P) = \{\, \mathsf{pile}(ab) \mid a \in \{0, \ldots, l\}, b \in \{0, \ldots, r\}, \left(\begin{array}{c} a{\geq}c \rightarrow b{\geq}d \wedge \\ a{\geq}e \rightarrow b{\geq}f \wedge \\ a{\geq}g \leftarrow b{\geq}h \wedge \\ a{\geq}i \leftarrow b{\geq}j \end{array}\right) \}$$

Let’s use a shorter notation for comparing 2CGs and their topologies:

$$\mathcal{O}\left(\begin{tikzcd}[row sep=small] & \_5 \arrow[d] \\ 4\_ \arrow[d] & \_4 \arrow[d] \arrow[dl] \\ 3\_ \arrow[d] \arrow[dr] & \_3 \arrow[d] \arrow[dl] \\ 2\_ \arrow[d] & \_2 \arrow[d] \\ 1\_ & \_1 \end{tikzcd}\right) = \begin{array}{cccccc} & & & 45 & & \\ & & 44 & & 35 & \\ & 43 & & 34 & & 25 \\ 42 & & 33 & & 24 & \\ & 32 & & 23 & & 14 \\ & & 22 & & 13 & \\ & 21 & & 12 & & 03 \\ 20 & & 11 & & 02 & \\ & 10 & & 01 & & \\ & & 00 & & & \end{array}$$

the arrows in R and L and the values of l and r are easy to read from the 2CG at the left, and we omit the ‘pile’s at the right.

In a situation like the above we say that the 2CG in the ‘$\mathcal{O}(\ldots)$’ *generates* the ZHA at the right. There is an easy way to draw the ZHA generated by a 2CG, and a simple way to find the 2CG that generates a given ZHA. To describe them we need two new concepts.

If (A, R) is a directed graph and $S \subset A$ then $\downarrow S$ is the smallest open set in $\mathcal{O}_R(A)$ that contains S. If (A, R) is a ZDAG with black pawns moves as its arrows, think that the ‘1’s in S are painted with a black paint that is very wet, and that that paint flows into the ‘0’s below; the result of $\downarrow S$ is what we get when all the ‘0’s below ‘1’s get painted black. For example: $\downarrow 0_0^0 0_0^1 = 0_1^0 1_1^1$. When (P, A) is a 2CG and $S \subseteq P$, we have to think that the paint flows along the arrows, even if some of the intercolumn arrows point upward. For example:

$$\downarrow\left(\begin{tikzcd}[sep=tiny] & 0 \\ 0 & 0 \\ 0 \arrow[ur] & 1 \arrow[l] \\ 1 & 0 \end{tikzcd}\right) = \left(\begin{tikzcd}[sep=tiny] & 0 \\ 0 & 1 \\ 1 \arrow[ur] & 1 \arrow[l] \\ 1 & 1 \end{tikzcd}\right)$$

and if S consists of a single point, $S = \{s\}$, then we may write $\downarrow s$ instead of $\downarrow\{s\} = \downarrow S$. In the 2CG above, we have (omitting the ‘pile’s):

$$\downarrow\_2 \;=\; \downarrow\{\_2\} = \downarrow\left(\begin{tikzcd}[sep=tiny] & 0 \\ 0 & 0 \\ 0 \arrow[ur] & 1 \arrow[l] \\ 0 & 0 \end{tikzcd}\right) = \left(\begin{tikzcd}[sep=tiny] & 0 \\ 0 & 1 \\ 1 \arrow[ur] & 1 \arrow[l] \\ 1 & 1 \end{tikzcd}\right) = 23, \quad \text{and} \quad \begin{array}{ll} & \downarrow\_4{=}24, \\ \downarrow 3\_{=}33, & \downarrow\_3{=}23, \\ \downarrow 2\_{=}23, & \downarrow\_2{=}23, \\ \downarrow 1\_{=}10, & \downarrow\_1{=}01, \end{array}$$

The second concept is this: the “generators” of a ZDAG D with white pawns moves as its arrows — or of a ZHA D — are the points of D that have exactly one white pawn move pointing *to* them (not *going out of* them).

If (P, A) is a 2CAG, then $\mathcal{O}_A(P)$ is a ZHA, and ‘$\downarrow$’ is a bijection from P to the generators of $\mathcal{O}_A(P)$; for example:

$$\mathcal{O}\left(\begin{array}{ccc} & & {}\_5 \\ & & \downarrow \\ 4\_ & & {}\_4 \\ \downarrow & & \downarrow \\ 3\_ & & {}\_3 \\ \downarrow & & \downarrow \\ 2\_ & & {}\_2 \\ \downarrow & & \downarrow \\ 1\_ & & {}\_1 \end{array}\right) = \begin{array}{cccccc} & & & 45 & & \\ & & 44 & & 35 & \\ & 43 & & 34 & & 25 \\ 42 & & 33 & & 24 & \\ & 32 & & 23 & & 14 \\ & & 22 & & 13 & \\ & 21 & & 12 & & 03 \\ 20 & & 11 & & 02 & \\ & 10 & & 01 & & \\ & & 00 & & & \end{array} \qquad \begin{array}{cccccc} & & & \cdot & & \\ & & \cdot & & \cdot & \\ & \cdot & & \cdot & & {}\_5 \\ 4\_ & & \cdot & & \cdot & \\ & 3\_ & & \cdot & & {}\_4 \\ & & \cdot & & \cdot & \\ & \cdot & & \cdot & & {}\_3 \\ 2\_ & & \cdot & & {}\_2 & \\ & 1\_ & & {}\_1 & & \\ & & \cdot & & & \end{array}$$

but if (P, A) is a 2CG with cycles, then $\mathcal{O}_A(P)$ is not a ZHA because each cycle generates a "gap" that disconnects the points of $\mathcal{O}_A(P)$. We just saw an example of a 2CG with a cycle in which $\downarrow 2\_ = 23 = \downarrow {}\_3 = \downarrow {}\_2$; look at its topology:

$$\mathcal{O}\left(\begin{array}{ccc} & & {}\_4 \\ & & \downarrow \\ 3\_ & & {}\_3 \\ \downarrow & \nearrow & \downarrow \\ 2\_ & \longleftarrow & {}\_2 \\ \downarrow & & \downarrow \\ 1\_ & & {}\_1 \end{array}\right) = \begin{array}{cccc} & & 34 & \\ & 33 & & 24 \\ & & 23 & \\ & & & \\ & & & \\ & 11 & & \\ 10 & & 01 & \\ & 00 & & \end{array}$$

16 Topologies as Heyting Algebras

The *open-set semantics* for Intuitionistic Propositional Logic is based on this idea: choose any topological space $(X, \mathcal{O}(X))$; the opens sets of $\mathcal{O}(X)$ will play the role of truth-values, and we define the components of a Heyting Algebra (sec.8) as this:

$$\begin{array}{rcll} \Omega & := & \mathcal{O}(X) & \\ P \le Q & := & P \subseteq Q & \\ \top & := & \{\, x \in X \mid \top \,\} & = \ X \\ \bot & := & \{\, x \in X \mid \bot \,\} & = \ \emptyset \\ P \wedge Q & := & \{\, x \in X \mid x \in P \wedge x \in Q \,\} & = \ P \cap Q \\ P \vee Q & := & \{\, x \in X \mid x \in P \vee x \in Q \,\} & = \ P \cup Q \\ P \overset{\mathrm{M}}{\to} Q & := & \{\, x \in X \mid x \in P \to x \in Q \,\} & \\ & = & \{\, x \in X \mid x \notin P \vee x \in Q \,\} & = \ (X \backslash P) \cup Q \end{array}$$

However, this '$\overset{\mathrm{M}}{\to}$' *may* return a non-open result even when given open inputs,

$$\begin{smallmatrix} & 0 & \\ 1 & & 0 \\ 1 & & 0 \end{smallmatrix} \overset{\mathrm{M}}{\to} \begin{smallmatrix} & 0 & \\ 0 & & 0 \\ 1 & & 1 \end{smallmatrix} \;=\; \begin{smallmatrix} & 1 & \\ 0 & & 1 \\ 1 & & 1 \end{smallmatrix}$$

so our definition is broken; we can fix it by taking the interior:

$$P \to Q \quad := \quad \mathsf{int}(P \overset{\mathrm{M}}{\to} Q) \quad = \quad \mathsf{int}((X \backslash P) \cup Q)$$

Theorem 16.1 *For any topological space $(X, \mathcal{O}(X))$ the structure $(\Omega, \leq, \top, \bot, \wedge, \vee, \to)$ defined as above is a Heyting Algebra. In particular, this holds for any $P, Q, R \in \Omega$: $P \leq (Q \to R)$ iff $(P \wedge Q) \leq R$.*

Proof. Standard; see for example [Awo06] (section 6.3). ■

Note that Theorem 16.1 gives us another way to calculate the connectives in 2CGs. In sec.7 we saw how to calculate $\neg\neg P \to P$ in a certain ZHA when $P = 10$; compare it with the "topological" way, in which the truth-values are subsets of $\begin{smallmatrix} & \bullet & \\ \bullet & & \bullet \\ \bullet & & \bullet \end{smallmatrix}$:

$$\underbrace{(\underbrace{\neg\underbrace{\neg\underbrace{P}_{10}}_{02}}_{20}) \to \underbrace{P}_{10}}_{12} \qquad\qquad \underbrace{(\underbrace{\neg\underbrace{\neg\underbrace{P}_{\begin{smallmatrix} & 0 & \\ 0 & & 0 \\ 1 & & 0 \end{smallmatrix}}}_{\begin{smallmatrix} & 0 & \\ 0 & & 1 \\ 0 & & 1 \end{smallmatrix}}}_{\begin{smallmatrix} & 0 & \\ 1 & & 0 \\ 1 & & 0 \end{smallmatrix}}) \to \underbrace{P}_{\begin{smallmatrix} & 0 & \\ 0 & & 0 \\ 1 & & 0 \end{smallmatrix}}}_{\begin{smallmatrix} & 0 & \\ 0 & & 1 \\ 1 & & 1 \end{smallmatrix}}$$

17 Converting between ZHAs and 2CAGs

Let's now see how to start from a 2CAG and produce its topology (a ZHA) quickly, and how to find quickly the 2CAG that generates a given ZHA.

From 2CAGs to ZHAs. Let $(P, A) = \mathsf{2CG}(l, r, R, L)$ be a 2CAG, and call the ZHA generated by it H. Then the top point of H is lr, and its bottom point is 00. Let $C := \{00, \downarrow 1_{\_}, \downarrow 2_{\_}, \ldots, \downarrow l_{\_}, lr\}$, i.e., the left generators (see the end of sec.15) plus $\bot$ and $\top$; then C has some of the points of the left wall (sec.4) of H, but usually not all. To "complete" C, apply this operation repeatedly: if $ab \in C$ and $ab \neq lr$, then test if either $(a+1)b$ or $a(b+1)$ are in C; if none of them are, add $a(b+1)$, which is northeast of ab. When there is nothing else to add, then C is the whole of the left wall of H. For the right wall, start with $D := \{00, \downarrow\_1, \downarrow\_2, \ldots, \downarrow\_r, lr\}$, and for each $ab \in C$ with $ab \neq lr$, test if either $(a+1)b$ or $a(b+1)$ are in D; if none of them are, add $(a+1)b$, which is northwest of ab. When there is nothing else to add, then D is the whole of the right wall of H.

In the acyclic example of the last section this yields:

$$\begin{array}{rcl}
C & = & \{00, \downarrow 1_, \downarrow 2_, \downarrow 3_, \downarrow 4_, lr\} \\
 & = & \{00, 10, 20, 32, 42, 45\} \\
 & \rightsquigarrow & \{00, 10, 20, 21, 22, 32, 42, 43, 44, 45\}, \\
D & = & \{00, \downarrow_1, \downarrow_2, \downarrow_3, \downarrow_4, \downarrow_5, lr\} \\
 & = & \{00, 01, 02, 03, 14, 25, 45\} \\
 & \rightsquigarrow & \{00, 01, 02, 03, 13, 14, 24, 25, 35, 45\}.
\end{array}$$

and the ZHA is everything between the "left wall" C and the "right wall" D.

From ZHAs to 2CAGs. Let H be a ZHA and let lr be its top point. Form the sequence of its left wall generators (the generators of H in which the arrow pointing to them points northwest) and the sequence of its right wall generators (the generators of H in which the arrow pointing to them points northeast). Look at where there are "gaps" in these sequences; each gap in the left wall generators becomes an intercolumn arrow going right, and each gap in the right wall generators becomes an intercolun arrow going left. In the acyclic example of the last section, this yields:

$$\begin{array}{llll}
 & & _5 = 25 & \\
 & & & \text{(gap becomes } 2_ \leftarrow _5) \\
4_ = 42 & & _4 = 14 & \\
 & \text{(no gap)} & & \text{(gap becomes } 1_ \leftarrow _4) \\
3_ = 32 & & _3 = 03 & \\
 & \text{(gap becomes } 3_ \rightarrow _2) & & \text{(no gap)} \\
2_ = 20 & & _2 = 02 & \\
 & \text{(no gap)} & & \text{(no gap)} \\
1_ = 10 & & _1 = 01 &
\end{array}$$

We know l and r from the top point of the ZHA, and from the gaps we get R and L; the 2CAG that generates this ZHA is:

$$(4, 5, \left\{3\_ \rightarrow \_2\right\}, \left\{\begin{matrix} 2\_ \leftarrow \_5, \\ 1\_ \leftarrow \_4 \end{matrix}\right\}).$$

Theorem 17.1 *The two operations above are inverse to one another in the following sense. If we start with a ZHA H, produce its 2CAG, and produce a ZHA H' from that, we get the same ZHA: $H' = H$. In the other direction, if we start with a 2CAG $(P, A) = \mathsf{2CG}(l, r, R, L)$, produce its ZHA, H, and then obtain a 2CAG $(P', A') = \mathsf{2CG}(l', r', R', L')$ from H, we get back the original 2CAG if and only if it didn't have any superfluous arrows; if the original 2CAG had superflous arrows then then new 2CAG will have $l' = l$, $r' = r$, and R' and L' will be R and L minus these "superfluous arrows", that are the ones that can be deleted without changing which 2-piles are forbidden. For example:*

$$
\left(\begin{array}{ccc}
4_{\_} & \longrightarrow & {}_{\_}4 \\
\downarrow & \searrow & \downarrow \\
3_{\_} & & {}_{\_}3 \\
\downarrow & \searrow & \downarrow \\
2_{\_} & \longrightarrow & {}_{\_}2 \\
\downarrow & & \downarrow \\
1_{\_} & & {}_{\_}1
\end{array}\right)
\quad\rightsquigarrow\quad
\begin{array}{cccccc}
 & 44 & & & & \\
 & & 34 & & & \\
 & 33 & & 24 & & \\
32 & & 23 & & 14 & \\
 & 22 & & 13 & & 04 \\
 & & 12 & & 03 & \\
 & 11 & & 02 & & \\
10 & & 01 & & & \\
 & 00 & & & &
\end{array}
\quad\rightsquigarrow\quad
\left(\begin{array}{ccc}
4_{\_} & \longrightarrow & {}_{\_}4 \\
\downarrow & & \downarrow \\
3_{\_} & & {}_{\_}3 \\
\downarrow & & \downarrow \\
2_{\_} & \longrightarrow & {}_{\_}2 \\
\downarrow & & \downarrow \\
1_{\_} & & {}_{\_}1
\end{array}\right)
$$

In this case we have $R = \left\{\begin{array}{c} 4_{\_}\to{}_{\_}4, \\ 4_{\_}\to{}_{\_}3, \\ 3_{\_}\to{}_{\_}2, \\ 2_{\_}\to{}_{\_}2 \end{array}\right\}$ *and* $R' = \left\{\begin{array}{c} 4_{\_}\to{}_{\_}4, \\ 2_{\_}\to{}_{\_}2 \end{array}\right\}$.

18 ZHA Logic is between IPL and CPL

In standard terminology, this is: ZHA Logic is a superintuitionistic logic ([CZ97], p.109) of "bounded width 2", i.e., where the axiom $\mathbf{BW}_2$ of [CZ97], p.112, holds. But let's see this in elementary terms.

Let S be this sentence:

$$
\begin{array}{rcl}
S_P & := & P \to (Q \vee R) \\
S_Q & := & Q \to (R \vee P) \\
S_R & := & R \to (P \vee Q) \\
S & := & S_P \vee S_Q \vee S_R
\end{array}
$$

S can't be an intuitionistic theorem because in this Heyting Algebra, with these values for P, Q, R,

$$
(W,A) = \begin{tikzcd}[column sep=small] & 1 \arrow[dl] \arrow[d] \arrow[dr] & \\ 2 & 3 & 4 \end{tikzcd}
\qquad
(\mathcal{O}_A(W), \subset_1) = \begin{tikzcd}[column sep=small]
& 1\overset{1}{1}1 & \\
& 1\overset{0}{1}1 \arrow[u] & \\
1\overset{0}{1}0 \arrow[ur] & 1\overset{0}{0}1 \arrow[u] & 0\overset{0}{1}1 \arrow[ul] \\
1\overset{0}{0}0 \arrow[u] \arrow[ur] & 0\overset{0}{1}0 \arrow[ul] \arrow[ur] & 0\overset{0}{0}1 \arrow[ul] \arrow[u] \\
& 0\overset{0}{0}0 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{array}{rcl} P & = & 1\overset{0}{0}0 \\ Q & = & 0\overset{0}{1}0 \\ R & = & 0\overset{0}{0}1 \end{array}
$$

we have $S = 1\overset{0}{1}1 \neq \top = 1\overset{1}{1}1$.

One way to define a *valuation* for a sentence S with variables $\mathsf{Vars}(S)$ — in our example we have $\mathsf{Vars}(S) = \{P, Q, R\}$) — is as a pair made of a Heyting Algebra H and a function $v : \mathsf{Vars}(S) \to H$. A looser definition is that a valuation for S is a pair made of 1) something that generates a Heyting Algebra in a known, canonical way, and 2) a function from $\mathsf{Vars}(S)$ to the elements of that HA. So:

A *classical valuation* for S is a valuation of the form $(\{0,1\}, v)$.

A *ZHA-valuation* for S is a valuation of the form (H, v), where H is a ZHA.

A *finite DAG-valuation* for S is a valuation of the form $((W, A), v)$, where W is a finite set and $A \subseteq W \times W$ is a set of arrows on W; the Heyting Algebra on $(W, \mathcal{O}_A(W))$ is built as in sec.16.

A *2CG-valuation* for S is a finite DAG-valuation for S of the form $((P, A), v)$, where (P, A) is a 2-column graph; each 2CG-valuation is naturally equivalent to a ZHA-valuation, and vice-versa.

A *classical countermodel* for S is classical valuation for S in which the value of S is not $\top$; a *ZHA-countermodel* for S is a ZHA-valuation for S in which the value of S is not $\top$; an *intuitionistic countermodel* for S is a finite DAG-valuation for S in which the value of S is not $\top$.

A sentence S is a *classical tautology* (notation: $S \in \mathsf{Taut}(\mathsf{CPL})$) if S has no classical countermodels; a sentence S is a *ZHA-tautology* (notation: $S \in \mathsf{Taut}(\mathsf{ZHAL})$); and a sentence S is an *intuitionistic tautology* (notation: $S \in \mathsf{Taut}(\mathsf{IPL})$) of S has no finite-DAG countermodels.

It is a standard result that the intuitionistic *theorems* are exactly the finite-DAG *tautologies*; this can be seen using Gödel translation (see [Göd86] and [Tro86]) to translate S to S4, and then using modal tableaux for S4 ([Fit72]) to look for a countermodel; in standard terminology, W is a set of "worlds", A

is an "accessibility relation" or a notion of which worlds are "ahead" of which other ones, and (W, A^*) is a Kripke frame for S4.

The sentence $S = S_P \vee S_Q \vee S_R$ of the beginning of the section is a good example for introducting tableau methods for modal logics to "children", as the tableau that it generates doesn't have branches. We can present the method directly and in elementary terms, as we will do now.

Fix a set W and a relation $A \subseteq W \times W$. We will say that β is "ahead" of α when $(\alpha, \beta) \in A^*$, i.e., when there is a path $\alpha \to \ldots \to \beta$ using only arrows in A. Let P and Q be open sets in $\mathcal{O}_A(W)$. The only way to have $P \vee Q$ false in a world α (notation: $(P \vee Q)_\alpha = 0$) is to have $P_\alpha = 0$ and $Q_\alpha = 0$. The only way to have $P \to Q$ false in a world α, i.e., $(P \to Q)_\alpha = 0$ is to have $P_\beta = 1$ and $Q_\beta = 0$ in *some* world β, with β ahead of α.

Let $((W, A), v)$ be a finite DAG-countermodel for $S = S_P \vee S_Q \vee S_R$. Then $v(P), v(Q), v(R) \in \mathcal{O}_A(W)$; we will omit the '$v$'s. If $((W, A), v)$ is a countermodel this means that $S \neq \top$, and there is some world α in W in which $S_\alpha = 0$. Fix this α. $S_\alpha = 0$ means $(S_P \vee S_Q \vee S_R)_\alpha = 0$, which means that $(S_P)_\alpha = 0$, $(S_Q)_\alpha = 0$, and $(S_R)_\alpha = 0$. $(S_P)_\alpha = 0$ means $(P \to (Q \vee R))_\alpha = 0$, which means that there is a world β ahead of α in which $P_\beta = 1$ and $(Q \vee R)_\beta = 0$, and $(Q \vee R)_\beta = 0$ means $Q_\beta = 0$ and $R_\beta = 0$; similarly, $(S_Q)_\alpha = 0$ means that there is a world γ ahead of α in which $Q_\gamma = 1$, $R_\gamma = 0$, $P_\gamma = 0$, and $(S_R)_\alpha = 0$ means that there is a world δ ahead of α in which $R_\delta = 1$, $P_\delta = 0$, $Q_\delta = 0$. In diagrams:

$$\begin{tikzcd} & \alpha \arrow[dl] \arrow[d] \arrow[dr] & \\ \beta & \gamma & \delta \end{tikzcd}$$

$$\begin{array}{ccc} & \begin{array}{c} S_\alpha = 0 \\ (S_P)_\alpha = (P \to (Q \vee R))_\alpha = 0 \\ (S_Q)_\alpha = (Q \to (R \vee P))_\alpha = 0 \\ (S_R)_\alpha = (R \to (P \vee Q))_\alpha = 0 \end{array} & \\ \swarrow & \downarrow & \searrow \\ \begin{array}{c} P_\beta = 1 \\ (Q \vee R)_\beta = 0 \\ Q_\beta = 0 \\ R_\beta = 0 \end{array} & \begin{array}{c} Q_\gamma = 1 \\ (R \vee P)_\gamma = 0 \\ R_\gamma = 0 \\ P_\gamma = 0 \end{array} & \begin{array}{c} R_\delta = 1 \\ (P \vee Q)_\delta = 0 \\ P_\delta = 0 \\ Q_\delta = 0 \end{array} \end{array}$$

Note that β and γ are "independent" in the sense that in A^* we can't have an arrow $\beta \to \gamma$ and neither an arrow $\gamma \to \beta$; we can't have $\beta \to \gamma$ because $P_\beta = 1$ but $P_\gamma = 0$, and we can't have $\gamma \to \beta$ because $Q_\gamma = 1$ but $Q_\beta = 0$. We can use a similar argument to show that γ and δ are independent, and to show also that δ and β are independent.

We can't have three independent points in a 2-column graph, so we have finite DAG-countermodels for S but no 2CG-countermodels for S, and so no

ZHA-countermodels for S. This means that S is not an intuitionistic tautology, but it is a ZHA-tautology. It is easy to see that $\mathsf{Taut}(\mathsf{IPL}) \subset \mathsf{Taut}(\mathsf{ZHAL}) \subset \mathsf{Taut}(\mathsf{CPL})$, and we saw that $S \notin \mathsf{Taut}(\mathsf{IPL})$, $S \in \mathsf{Taut}(\mathsf{ZHAL})$, $(\neg\neg P \to P) \notin \mathsf{Taut}(\mathsf{ZHAL})$, $(\neg\neg P \to P) \in \mathsf{Taut}(\mathsf{IPL})$, which means that:

$$\mathsf{Taut}(\mathsf{IPL}) \subsetneq \mathsf{Taut}(\mathsf{ZHAL}) \subsetneq \mathsf{Taut}(\mathsf{CPL})$$

and so "ZHA Logic", *which we have not defined via a deduction system,* only by the notions of "ZHA countermodels" and "ZHA tautologies", is strictly between Intuitionistic Logic and Classical Logic, and is different from both.

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Review: Guillermo E. Rosado Haddock's *Unorthodox Analytic Philosophy*

Texts in Philosophy, Volume 27, College Publications, 2018, 520pp., ISBN 978-1848902732

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The present book is a book of essays, all by Guillermo E. Rosado Haddock, retired Full Professor of Philosophy at the University of Puerto Rico "Río Piedras". It combines two sorts of writings: papers published in peer review journals or as chapters of books, in addition to three – essays (2), (4) and (7) – unpublished so far, and critical studies. The book is divided into three parts. The first, titled *Some Fundamental Issues*, consists of papers (1-6) concerned with philosophical themes: from the relations among philosophy, logic and mathematics to the semantic structure of reference, from the epistemic status of *a posteriori* statements to that, ontological, of concepts and the role of intuition. The second, *Husserl and other Philosophers* (papers 7-11), is concerned mostly with Husserl's views about logic and mathematics in comparison with those of some eminent philosophers like Kant, Frege and Carnap, in order to reveal differences and common origins, and with those of one great mathematician of the past: Riemann. Finally, the third part – *Doing Rigorous Philosophy* – contains critical studies of books (12-20), mostly on Frege's views, Husserl's influence on Carnap and Husserl's work on logic, and a critical commentary (21) of a long paper on naturalism by Kanitscheider.

The book is written by a philosopher devoted to rigorous analysis and methodology that honestly rejects the widespread and apparently simplistic division of philosophy in analytic versus continental: it is just a prejudice to think that this rejection would open "the doors to all sorts of irrationalisms and obscurantisms" (p. ix). Author's intention is clearly to trace the boundaries between orthodox approaches to analytic philosophy and "unorthodox", those that are not led by an ideologically blinded empiricism. The label 'unorthodox analytic philosophy' serves precisely the Author to point out that one cannot do serious philosophy without taking in account the development, at least, of the three more exact sciences – logic, mathematics and physics – but presupposing the "meta-dogma of empiricist ideology" (p. 1).

To be honest, Author's concerns have been here more with logic and mathematics, rather than physics, but still mathematics finds its own empiricism's ideological cousin in *nominalism*. Nominalism, albeit in different ways articulated, is the attempt to avoid accepting abstract entities that are not individuals – *qualia*, introduced by Nelson Goodman, are. Quine's "incorrect" – according to the Author – theses or criterion that 'to be is to be the value of a (first-order) variable' has served and still serves nominalism to justify its own principles about a flat ocean of individuals as structurally designed by first-order logic: yet, "if they accept the semantics of classical first-order logic, that is classical model theory, there is no way out of Platonism, since one is dealing with full blown mathematical structures" (p. 2). On the other hand, unorthodox analytic philosophy is essentially a contemporary version of rigorous philosophy, in its most genuine sense, whose Frege, Husserl, Popper and Whitehead, for different reasons, are just some of the most eminent – among the many, especially nowadays, analytic philosophers – key players, since they in no way accepted that meta-dogma.

After this general introduction to the spirit of the book, it seems to me time to deepen the contents presented and articulated by the Author. I'll come back to some considerations about Author's Platonist reading of Husserl's thought later. Since there is a lot of material here, for matter of space and opportunity my eyes will not linger on every topic if not briefly, with exception of the most relevant chapters, selected in order to provide a coherent and satisfactory review of the whole work.

The first chapter (or paper) gives the reader the adequate frame to read and interpret Author's investigations. It discusses the interplay between logic, mathematics and philosophy, as mentioned in the title. The Author offers various examples, from the application of group theory to semantics – "a clear case of the application of the mathematical theory of groups of transformations to philosophical semantics" (p. 17) – to the application of Husserlian philosophical semantics to logic where, f.i., it is well remarked how for Husserl not truth-values but, rather, *states of affairs* are the referents of statements. A choice, this one, "by far more informative than Frege's" (p. 16), where truth-values make statement to collapse. However, the core of the chapter is devoted to remark how the basic principle of nominalism is simply a meta-dogma, a meta-semantic criterion whose valid application is unjustified. Indeed, the Author proceeds, even confining themselves to first-order logic for avoiding any commitment further to that to mere individuals (abstract entities), nominalists have to do with both structures in a pure infinitary (to be read "non constructive") way and mathematical objects: the classic examples are the Upward Löwenheim-Skolem Theorem and the Robinson's Model Completeness Test. In few words, the interplay between the limitation to first-order theories and those theorems simply forces us to "accept classical, that is, first-order model theory" but not to refute abstract entities. It is shown that by the former theorem, infinitary non-denumerable structures are proved to exists and that, by the latter, true existential statements on mathematical entities cannot be *a priori* refuted. Hence, the acceptance of model theory – the acceptance, that is, of the possibility for logic of speaking about mathematics – turns Quine's criterion to a meta-dogma.

The Fine Structure of Sense-Referent Semantics (*An Excursus into Semantic and Mathematical Platonism*) is the first of the three papers published here for the first time. It is concerned with a deep introduction to and an application of Husserl's distinction between sense and referent – that distinction has been erroneously and unjustly attributed only to Frege, as the Author motivates in the introduction. Author's attention falls on statements and not of concepts words, just sketched. Such a distinction is articulated between modes of presentation of states of affair (sense) and states of affair or situation of affairs – roughly, equivalence classes of the states of affairs – (reference). In order to see the differences between the two semantic theories, the Author suggests to define a notion of *depth* of a semantic theory of sense and referent:

> [T]he depth of a sense-referent semantic theory is the number of semantic levels between the sense of a statement and its truth-value. In virtue of that definition, it follows that Frege's choice of a sense-referent theory has depth 0, whereas a semantic theory that has states of affairs as the referents of statements, but does not take into consideration the situations of affairs, has depth 1, and a theory like Husserl's, that considers both states of affairs and situations of affairs as intermediate levels between the level of senses and that of truth-values has depth 2. For semantic theories of natural languages that seems certainly enough. (p. 36)

An example of the application of the theory is to logic, in the case of first-order propositional functions: they have to be conceived as schemes of states of affairs. The application to mathematics is restricted to mathematical statements (theorems) and generates specific referents: sets of states of affair or abstract states of affair. The paper, then, proceeds about dual statements in mathematics and logic, on some meta-logical considerations and accounting for the case of Platonism in semantic theories.

The third paper, according to the Author, is "the most ambitious" (p. 7) and, thus, I dedicate to it some attention, all the more so that the fourth essay "On Analyticity *a posteriori* and Syntheticity *a priori*" – published here for the first time – "is a sort of less rigorous presentation of the results in the former" (*Ibid.*). This is a paper of epistemic value, likewise (4), that wonder whether analytic *a posteriori* statements are possible. In order to provide an answer, a new definition of analyticity – a refinement of Husserl's – is presented. The new definition of analyticity proposed seems to well survives criticisms to Kant's, Frege's and Carnap's definitions and, even, to that to Husserl's one. If Kant's definition is twofold, i.e., (a) as statement whose concept of its predicate is included in the concept of its subject and/or (b) as derivable from the Principle of Non-Contradiction, only (b) survives Quine's criticism of *Two Dogmas.* Inspired to (b), the unfortunate Frege's definition, as statements that can be derived from logical principles and definitions, does not survive "the collapse of logicism" (p. 59). Husserl's seems to be "more solid" (*Ibid.*), then: a statement is analytic if it is true and its truth can be completely formalized *salva veritate.* Author's attempt here is to overcome the unavoidable weakness of the Husserlian definition, traceable in its profound adequacy as a notion of logical truth. With respect to previous attempts (refinement of Husserl's), where the Author formulated unsatisfactory definitions, the new one adds an extra condition (iii).

> A statement σ is analytic if and only if: (i) $\{\sigma\}$ has a model M, (ii) if $\{\sigma\}$ has a model M, then any structure M^* isomorphic to M is also a model of $\{\sigma\}$, and (iii) $\{\sigma\}$ does not imply or presuppose the existence either of a physical world or of a world of consciousness. (p. 61)

Notwithstanding this definition, the answer to the question made in the title of the paper is, in a strict sense, 'no'. Indeed, it is argued for the existence of analytic statements just as instantiations of analytic laws (or constant-free statements). Those are what Husserl called "analytic necessities": statements with constants obtained by quantifier elimination from analytic laws. Since they are instantiations, they are *a posteriori.* Such instantiations of analytic laws, the Author argues however, do not satisfy (iii) above, being that constants occurring in them not necessarily mathematical constants.

Essay (5) and (6) deal with different issues in unorthodox analytic philosophy, already touched in the first essay. In particular, the former – *Some*

Heterodox Analytic Philosopher – presents a deepening of the argument about Robinson's Model-Completeness test in first-order model theory and its role to refute nominalism. The latter, the last paper of this first part of the book, serves instead the Author as a sort of transition to the second group touching an issue treated also in (7), namely, a critique of one of Kant's arguments occurring in the Transcendental Aesthetic of the first *Critique* whose purpose was to show that space (and time also) is not a concept but an intuition. As clearly pointed out in the Abstract, "[i]t is here shown that Kant's conclusion is completely unfounded, since one can reproduce those arguments on the basis both of the concept of a continuous manifold in Riemann's sense and of that of an extensive whole in Husserl's sense" (p. 105). Thus, reference to Riemann's views on the nature of (geometrical and empirical) space are fundamental. Finally, some remarks on Frege's and Husserl's divergences on the notion of whole are presented and discussed.

Here the second part of the volume begins. I will focus much more attention to essays (7) and (8) for different reasons: for the former is one of the unpublished paper so far and for the latter is concerned with the many reasons why Husserl should be considered an analytic but, clearly, unorthodox philosopher.

It is immediately apparent the topic of chapter (7), titled *Husserl and Kant: voilà la différence*. Too many times philosophers, both analytic and continental, did not clearly recognized rightly both convergences and divergences between the two. Certainly, both Kant and Husserl used the adjective 'transcendental' in naming their philosophy or philosophical approaches on the "common interest in putting the 'transcendental subject' at the centre stage of philosophical research and examining the conditions of possibility of having (scientific) knowledge" (p. 115). This approach becomes mature since Descartes but, according to the Author, if for Kant surely the theory of knowledge was first philosophy, on the other hand Husserl's thought cannot be confined in his transcendental phenomenology, clearly immersed in that tradition. In his course on old and new logic (1908-1909) – after the transcendental turn – Husserl clearly emphasized that it is *philosophical logic* which deserves the name of first philosophy – this anticipates, in a sense, what is argued in the next essay about the analytic character of Husserl's philosophical investigations. After a recall on Kant's theoretical philosophy and an introduction to Husserl's thought prior to his phenomenology, the Author reveals the main difference. The section *On Husserl on Logic and Mathematics* highlights Husserl's conception of mathematics and how it is distant in many points from that of Kant. Take some: (i) mathematics is a "formal ontology", "which is basically a conception of mathematics as a theory of structures" (p. 130), against Kant's purely phenomenical and constructive view of mathematical entities; (ii) mathematics "can be seen as a generalization of Riemann's conception of mathematics as a theory of manifolds" (*Ibid.*), that clearly contrasts with Kant's own reductionist conception; (i) and (ii) are then strictly related to the fact that (iii) mathematics is not based on one single fundamental mathematical concept, given that Husserl "acknowledged a plurality of fundamental concepts, which he called 'formal ontological categories'" (*Ibid.*). Furthermore and quite obviously, Husserl's conception of mathematics influenced his own view about the nature of intuition, again relevantly divergent from Kant's. Should be sufficient here to underline that two forms of intuition are possible, according to Husserl: the *eidetic* intuition or intuition of essences – and essences were banned from Kant's system – and the *categorical* intuition by which "even in our most simple 'sensible' perceptions there are categorial components – purely formal and intellectual for Kant – namely, states of affairs, relations, sets, etc." (p. 136) further to space and time.

With the eight paper, the Author feels the reader ready to see in which sense Husserl's many contributions were literally "ignored" by analytic philosophers, from to the philosophy of logic and mathematics to the epistemology of mathematics, passing through the philosophy of language and even the philosophy of physical science. The *incipit* is written in a way that could seem to be almost shocking to orthodox-friendly readers. The Author reports that in his work *Old and New Logic: Lessons 1908-1909*, Husserl considers philosophical logic – in the wide sense of logical analysis of concepts, language and even their philosophical implications – as "the presupposition and foundation for all the other genuine philosophical disciplines"; that it is, according to Husserl himself, "first philosophy'"; and even that in his philosophical investigation Husserl wants to proceed "analitically" (p. 145). Then, the paper comes back on the sense and reference distinction in Frege and Husserl accounting, first, for their independent 'discovery' – never accepted by analytic philosophers, yet by Frege – and, second, for the different semantic approaches to it, as I already mentioned. By the third section the Author writes about Husserl's refutation of psychologism in logic. An argumentation, this, judged properly analytic for it does not presuppose any phenomenological thesis and far superior in details and organization to Frege's one. The fourth section is, instead, dedicated to what Husserl thought about physical theories. The Author notices here the realist character of Husserl's conceptualist view on such theories, as they are based on *hypotheses cum fundamento in re* (p. 154):

> There is, thus, an ontological connection between objects (or concepts) that is objective and serves as a base for the building of even the most primitive sciences. (p. 153)

The sixth section is about Husserl on logic and mathematics. The great distance from logicists (Frege) consists in that, contrary to them, the two are parallel and not reducible disciplines: if "logic is essentially a syntactic-semantic discipline", on the other hand, "mathematics is more ontologically committed" (p. 157). The paper ends with Husserl's response to a letter from Frege (of 1906) and with a very brief appendix on his work *Philosophie der Aritmetik*.

I will not spend to too many words on the next three essays for almost all their contents have already been sketched writing about the previous items.

The ninth paper is concerned with Husserl's influence on Carnap. It is an issue on which the Author wrote a lot: a whole book *The Young Carnap's Unknown Master* (published in 2007), a paper – 'On the Interpretation of the Young Carnap's Philosophy' – included in a precedent collection of essays (*Against the Current*, published in 2013), and a critical study of Carnap's doctoral dissertation *Der Raum* (1922) – essay (15). This is a delicate issue, since the Author acknowledges strong influence of Husserl and the intellectual dishonesty of Carnap, telling us he knows of Carnap's appropriation of Husserl's ideas both as of Carnap's distinction (in *Logische Syntax*) between formation rules and transformation rules, mentioned in section 5 and 6 of (8), and as of the constitution of the heteropsychological in Carnap's *Aufbau*.

Husserl and Riemann is a paper on the influence of the great mathematician on Husserl. The importance of this topic to understand the origin of Husserl's views on mathematics as a formal ontology (a theory of formal structures) has been reported several times here, and appears in some details in section 5 of (1), 4 of (6), 4 an 5 of (7) as well as 3 of (8). After briefly showing that Frege had almost no influence on Husserl's views on logic and mathematics as well as on the sense-referent distinction, the Author argues that Husserl's conception of mathematics as a theory of structures and/or of manifolds is a direct generalization of Riemann's notion of manifold and that his views on physical geometry (empirical space) came directly from Riemann's reflections, as attested by letters of 1892 to Brentano, and of one 1897 and one another of 1901 to Natorp.

Finally, the eleventh and last paper of this second part, is concerned with with Husserl's contributions to the nature of mathematical knowledge or, better, with his epistemology of mathematics. Such contributions are clearly opposed to both the naturalist and the empiricist approaches, but even to pragmatist tendencies. In particular, the paper introduces to such mainstream tendencies and argues that they fail to distinguish the historical problem of the origin and evolution of mathematical knowledge from the epistemological one.

The third part is made of ten critical studies, nine of books and one, the last, of a paper. The choice of including critical studies in this collection is due to their usefulness "in polishing and developing" (p. ix) Author's views and as tools and models of analysis in making philosophy.

The first two critical studies are very extensive (about 30 pp. each) and about collections of papers both on Frege's works and philosophy and both edited by the Fregean scholar Matthias Schirn. If the Author confesses "that in the later years I have felt disappointed with the popularity that some erroneous and sometimes almost crazy interpretations of Frege have had in the Anglo American world" (p. 9), this has to be restricted, in particular, with respect exactly six papers of the second volume of the first collection, considered a "sample of the great prevailing confusion on how to interpret both Frege's view from 1879 and especially his view of 1892" (p. 237). For what concerns the second collection, mostly on Frege's philosophy of mathematics, the Author is convinced that will be regarded as an "irreplaceable reading", providing a "profound understanding of Frege's contributions to philosophy", including "some of its weaknesses" (p. 267).

The review of Oswaldo Chateaubriand's *Logical Forms* is the third critical study we find. It is a two-volumes book of philosophical analysis of truth and description and of logic, language and knowledge. No doubt that it would deserve to be much better known. Here, the Author focuses in particular on the discussion of Chateaubriand's criticism to Quine and the characterization of logical truth. Furthermore, though Chateaubriand seems not to be acquainted, many affinities between some of his views on logical and semantic issues and those of Husserl are noticed.

The next two studies are both concerned with Carnap's works and ideas in a twofold way. Item (15) offers a brief exposition Carnap's "not well known and not well understood" (p. 9) doctoral theses *Der Raum*. The Author sees here an apparent influence of Husserl on that work and tries "to correct some misleading renderings of that work" (p. 327). In particular, Carnap's defence of the synthetic *a priori* is clearly but much nearer to Husserl's views rather than to Kant's. The Author leaves, then, the reader with a question that "does not seem to have an answer: Did Carnap discuss this issue with Husserl during the years 1919 to 1921?" (p. 356). The fifth critical study, instead, analyses a collection of essays on Carnap edited by Cirera, Ibarra and Mormann. The Author critically assesses the several renderings of Carnap while sharpening his own interpretations. Further, he presents criticisms to some consequences of the demise of neopositivism.

The next four (17-20) critical studies have various subjects. Item (17) concerns a publication by Elisabeth Schuhmann of lectures of Husserl on old and new logic of 1908-1909 – after his transcendental turn. In this review the Author will concentrate on a few issues that will better understand Husserl's relation to analytic philosophy in general. There, Husserl stresses that philosophical logic is first philosophy and not transcendental phenomenology, as discussed in more details in (7). Worth noting the fact that Husserl answers critically to Frege's "confusion" – in his letters to Husserl of 1906 – of identifying what is *having the same sense* with what is *being logically equivalent.* The review of the book *Lógica, Matemáticas y Realidad* by Anastasio Alemán is essay (18). It touches each issue included, but is mainly concerned with the papers more directly related to philosophy of mathematics (papers 1-3 and 5). The eight critical study, is of a collection edited by Jaakko Hintikka and titled *From Dedekind to Gödel* that, contrary to most collections on the philosophy of mathematics, allows not so well known interpreters to have space. This explains why the review itself is titled *The Other Philosophers of Mathematics*: the term 'other' is used in the sense of 'non-Frege'. The Author feels particularly interested, even if he judges as "unbalanced" (p. 419), the combination of contributions and their contents. The last study (20) is an assessment of a book of essays edited by the Fregean scholar Dirk Greimann *Essays on Frege's Conception of Truth.* It consists of nine papers, some of which are only marginally concerned with Frege's views about truth, even for Frege wrote relatively little on that issue. According to the Author, it "can serve as perfect examples of the reinterpretation of Frege's views" (p. 445) on truth. At the same time the Author feels not very satisfied when he comments that "[i]n general, this collection of papers is not especially illuminating. I have opted to say very little about some of the papers, in order to concentrate my efforts on pointing to some weaknesses of a few of the most questionable ones" (p. 446).

Finally, a commentary to a long paper by Bernulf Kanitscheider, published in *Erwägen Wissen Ethik*, now extinct. The format of the journal was very similar to that of *The Library of Living Philosophers* but, here, a philosopher wrote a long paper and some twenty scholars criticized him. Then, the original author responded to their criticisms. What the Author sees relevant of this reading is that it clearly arises how Kanitscheider is assuming as valid Quine's views on naturalized epistemology as well as those on ontology: "a naturalism for which the unity of nature can be considered as the guiding idea" (p. 462).

Let me spend a few words on Author's work on Husserl, Husserl conception of mathematics.

Regarding Husserl in particular, readers who are looking for an analysis of Husserl's views on transcendental subjectivity, intentionality or consciousness will not be satisfied. The focus of the book is clearly other than the structures of transcendental subjectivity and its faculties or operations, that occupied Husserl in his distinctively phenomenological writings. The emphasis on Husserl's unorthodox approach to philosophical analysis and theoretical constructions stresses the role of philosophical logic, rather than of transcendental logic, as first philosophy. From this, it arises quite natural to interpret Husserl's transcendental phenomenological turn as a matter of mere methodology, a philosophical device for conceptual analysis. A consequence may be, then, the conclusion that a Platonist tendency of realism survives his transcendental turn.

At the same time, the issue of whether Husserl is a realist or an idealist (conceptualist) about mathematics and mathematical entities may not seem to be plainly clear to some readers. Recall what the Author notices about the realist foundation of Husserl's conceptualism about physical theories, as they are based on *hypotheses cum fundamento in re* (p. 154). Such a notion is introduced recalling that such theories "go far above the realm of [empirical] induction". According to this passage, physical theories and mathematics, then, seem to have (partially) overlapping domains, at least with respect to those regions of physics being not linked to empirical induction. Thus, it seems plausible to extend Husserl's view up to his conception of mathematics and mathematical entities conceived as (infinitary) structures. In this case, mathematical structures would have *fundamentum in re*. In this case, the relation between philosophical and transcendental logics discussed above would lead the reader of Husserl to imagine his idealism as a form of epistemic Platonist structuralism, despite the constructivist bias often associated to his epistemic view (or phenomenology) and surely due to the impact and affection of the phenomenological turn.

In particular, an answer to the issue of whether Husserl's kind of idealism can be articulated as a form of constructivism about mathematics (i.e., a predicative view) or as a form of mathematics more similar to that of Frege (i.e., an impredicative one) may find out an answer looking to the notions of intuition discussed in (7). Author's position seems to be that Husserl had strong Platonist tendencies and I personally agree with him. Both forms of intuitions, the eidetic and the categorical, lead the transcendental subject to the knowledge of the *essence* and of the *formal structure* constituting the essence itself – states of affairs, relations, sets, etc. – respectively. It is a fact. But this might, then, give back a non reductionist but Platonist form of realism about mathematics: a formal ontology. This hardly can be seen as a form of constructivism.

Notwithstanding these issues, *Unorthodox Analytic Philosophy* can be highly recommended for its perspective on several issues in analytic philosophy. Husserl's contributions on foundational issues in logic, mathematics and the exact sciences have been long neglected without concrete rationales. I found it a pleasure to read this collection of essays. They are clearly written and thought-provoking, especially those (analytic) philosopher who, even nowadays, knows almost nothing about the origin of Husserl's theoretical visions on formal ontology and his contributions on logic and mathematics. The book covers an interesting range of topics in a vibrant and harmonic sound. Rosado Haddock is also, unlike most analytic philosophers, a Platonist about logic and mathematics, but it comes quite natural to be in the suffocating season of philosophy affected by the orthodox analytic influence. Hopes and efforts have to be oriented to end that epoch of philosophy.

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