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Forcing and Ultrafilters on FIN^∞

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Abstract

We study a forcing notion analogous to Mathias forcing defined in the space of infinite block sequences of finite sets of natural numbers.

Keywords: Ramsey ultrafilters, strongly summable ultrafilters, Mathias forcing, block sequences of finite sets.

Introduction

In this article we study certain classes of ultrafilters on the set of natural numbers that are linked to combinatorial theorems in the realm of Ramsey theory. We pay particular interest to some related forcing notions. The space FIN^∞ of block sequences of finite sets of natural numbers is of central importance in this study.

We first recall the definition of the Ramsey property for subsets of the space $[\mathbb{N}]^\infty$ of all infinite sets of natural numbers. With the product topology (the topology inherited from the product topology on $2^{\mathbb{N}}$), this space is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$, the irrational numbers. The exponential topology of this space, also called the Ellentuck topology, is finer than the product topology and it is generated by the basic sets of the form

$$[a, A] = \{X \in [\mathbb{N}]^\infty : a \sqsubset X \subseteq A\},$$

where a is a finite set of natural numbers, A is an infinite subset of $\mathbb{N}$, and $a \sqsubset X$ means that a is an initial segment of X in its increasing order.

We say that a subset $\mathcal{A} \subseteq [\mathbb{N}]^\infty$ is Ramsey, or has the Ramsey property, if for every $[a, A]$ there is an infinite subset B of A such that $[a, B] \subseteq \mathcal{A}$ or $[a, B] \cap \mathcal{A} = \emptyset$. Silver proved that all analytic subsets of $[\mathbb{N}]^\infty$ have the Ramsey property. His proof has a metamathematical character, as opposed to the combinatorial proof of Galvin and Prikry for the Borel sets. Ellentuck [5] gave a topological proof of Silver's result by showing that a subset of $[\mathbb{N}]^\infty$ is Ramsey if and only if it has the property of Baire with respect to the exponential topology. Similar results have been obtained for the space FIN^∞ of infinite block sequences of finite sets of natural numbers (see [12]).

Using the axiom of choice a set that is not Ramsey can be found, but Mathias, in [10], shows that it is consistent with ZF+DC (Zermelo-Fraenkel set theory with the axiom of dependent choices) that every subset of $[\mathbb{N}]^\infty$ is Ramsey, provided that ZFC is consistent with the existence of an inaccessible cardinal. For this, Mathias introduced a forcing notion that we will describe below. Here, we study some forcing notions related to the Ramsey theory of FIN^∞.

García Ávila makes a comparative study of these and other forcing notions in [7], and in particular, a forcing notion analogous to Mathias forcing adapted to the space FIN^∞. She proves that this notion has a pure decision property (a Prikry property) and asks if it has a property analogous to the Mathias property that an infinite subset of a Mathias generic real is also a Mathias generic real. We answer this question positively.

We use standard set theoretic notation. The set of natural numbers is called indistinctly $\mathbb{N}$ and ω. Each natural number n is identified with the set of its predecessors $\{0, \dots, n-1\}$. If X is a set, $[X]^n$ is the collection of subsets of X with exactly n elements; $[X]^\infty$ is the collection of infinite subsets of X. $[X]^{<\infty} = \bigcup_{n\in\omega}[X]^n$ is the set of finite subsets of X.

1 Ultrafilters on $\mathbb{N}$ related to theorems of Ramsey and Hindman

We start recalling definitions of certain types of ultrafilters on the set of natural numbers $\mathbb{N}$. In particular we will consider selective ultrafilters and strongly summable ultafilters. Selective ultrafilters on $\mathbb{N}$ are related to Ramsey's theorem, and strongly summable ultrafilters, to Hindman's theorem on finite sums. Union ultrafilters play a similar rôle with respect to the finite unions version of Hindman's theorem. These ultrafilters are studied in [1, 2]. Forcing notions connected to these types of ultrafiters have been considered in [4, 7, 10].

As usual, $\beta\mathbb{N}$ is the set of ultrafilters on $\mathbb{N}$. Identifying each natural number with the principal ultrafilter it generates, $\beta\mathbb{N} \setminus \mathbb{N}$ denotes the collection of non-principal ultrafilters on $\mathbb{N}$.

Definition 1 *Let $(n_i)_{i\in\omega}$ be an infinite strictly increasing sequence of positive integers.*

$$FS((n_i)) := \left\{ \sum_{i\in F} n_i : \emptyset \neq F \in [\mathbb{N}]^{<\infty} \right\}.$$

Definition 2 *Let $\mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}$. Then,* *(1) $\mathcal{U}$ is a P-**point** if for every partition $\mathbb{N} = \bigcup_{n<\omega} A_n$ into subsets that are not in $\mathcal{U}$, there is $B \in \mathcal{U}$ such that $|A_n \cap B| < \omega$ for every $n \in \omega$.*

*(2) $\mathcal{U}$ is a Q-**point** if for every partition $\mathbb{N}$ into finite sets $\{A_n : n \in \omega\}$, there is $B \in \mathcal{U}$ such that $|A_n \cap B| \leq 1$.*

*(3) $\mathcal{U}$ is **selective** if for every partition $\mathbb{N} = \bigcup_{n\in\omega} A_n$ where no A_n is element of $\mathcal{U}$, there is $B \in \mathcal{U}$ such that $|A_n \cap B| = 1$ for every $n \in \omega$.*

*(4) $\mathcal{U}$ is **strongly summable** if for every $A \in \mathcal{U}$ there is an infinite set $\{n_k : k \in \omega\}$ of positive integers such that $FS((n_k)) \in \mathcal{U}$ and $FS((n_k)) \subseteq A$.*

It is easy to verify that an ultrafilter $\mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}$ is selective if and only if it is a P-point and a Q-point.

We state now the following theorems of Ramsey and Hindman.

Theorem 1 *(Ramsey) Given positive integers n, r, for every $c : [\mathbb{N}]^n \to r$ there is an infinite set $H \subseteq \mathbb{N}$ such that c is constant on $[H]^n$.*

Such a set H is said to be homogeneous for c.

Theorem 2 *(Hindman) Let r be a positive integer. For every $c : \mathbb{N} \setminus \{0\} \to r$ there is an infinite strictly increasing sequence $(n_i)_{i\in\omega}$ of positive integers such that c is constant on $FS((n_i))$.*

The following propositions show how the types of ultrafilters defined above are related to these theorems.

Proposition 1 *Let $\mathcal{U} \in \beta\mathbb{N}\setminus\mathbb{N}$. Then $\mathcal{U}$ is selective if and only if for every $c : [\mathbb{N}]^2 \to 2$ there is $H \in \mathcal{U}$ such that c is constant on $[H]^2$.*

It is customary to refer to the function c as a coloring, and to say that $[H]^2$ is monochromatic for c.

Proposition 2 *Let $\mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}$. Then $\mathcal{U}$ is strongly summable if and only if for every coloring $c : \mathbb{N} \setminus \{0\} \to 2$ there is an infinite $B \subseteq \mathbb{N}$ such that $FS(B)$ is monochromatic and $FS(B) \in \mathcal{U}$.*

2 Ultrafilters on the set of finite subsets of $\mathbb{N}$.

Let $\mathrm{FIN} = [\mathbb{N}]^{<\infty} \setminus \{\emptyset\}$ be the collection of non-empty finite sets of natural numbers. By FIN^{∞} we denote the collection of all block sequences of elements of FIN, that is to say, all infinite sequences $a_0, a_1 \ldots$ where, for all $i \in \omega$, $a_i \in \mathrm{FIN}$ and $\max(a_i) < \min(a_{i+1})$. $\mathrm{FIN}^{<\infty}$ denotes the collection of finite block sequences of elements of FIN.

If $(s_i)_{i\in\omega}$ is a collection of pairwise disjoint elements of FIN,

$$\mathrm{FU}((s_i)) := \left\{ \bigcup_{i\in F} s_i : F \in \mathrm{FIN} \right\}.$$

Definition 3 *An utrafilter $\mathcal{U}$ on FIN is said to be a union ultrafilter if for every $A \in \mathcal{U}$ there is a pairwise disjoint infinite subset $\{a_n : n \in \omega\}$ of FIN such that $FU(\{a_n\}) \in \mathcal{U}$ and $FU(\{a_n\}) \subseteq A$.*

To motivate this definition we state now a version of Hindman's theorem in terms of finite unions. This gives also some intuition about the relation between union ultrafilters on FIN and strongly summable ultrafilters on $\mathbb{N}$

Theorem 3 (Hindman's theorem for unions) *Let $c : FIN \to 2$, then there exists an infinite collection $\{a_n : n \in \omega\}$ of pairwise disjoint elements of FIN such that $FU(\{a_n\})$ is monochromatic for c.*

The set $\{a_n : n \in \omega\}$ in the conclusion of Hindman's theorem for unions can be required to be in block position.

Definition 4 *An ultrafilter $\mathcal{U}$ on FIN is an ordered-union ultrafilter if for every $A \in \mathcal{U}$ there is a block sequence $\{a_n : n \in \omega\} \subseteq [\mathbb{N}]^{<\infty}$ such that $FU(\{a_n\}) \in \mathcal{U}$ and $FU(\{a_n\}) \subseteq A$.*

Let $\max : \mathrm{FIN} \to \mathbb{N}$ be the function that sends each finite set $s \in \mathrm{FIN}$ to its maximal element. If $\mathcal{U}$ is an ultrafilter on FIN, we call $\max(\mathcal{U})$ the ultrafilter on $\mathbb{N}$ given by $\{A \subseteq \mathbb{N} : \max^{-1}(A) \in \mathcal{U}\}$.

Theorem 4 ([1, 2]) *If $\mathcal{U}$ is a union ultrafilter on FIN, then $max(\mathcal{U})$ is a P-point. If $\mathcal{U}$ is an ordered-union ultrafilter on FIN, then $max(\mathcal{U})$ is also a Q-point, and therefore selective.*

This shows in particular that the existence of union ultrafilters and of ordered-union ultrafilters cannot be proved in ZFC. Nevertheless, their existence follows from CH or Martin's axiom.

Definition 5 *[2] Given an ultrafilter $\mathcal{U} \in \beta FIN$ and an ultrafilter $\mathcal{V} \in \beta\mathbb{N}$ we say that they are additively isomorphic if there is $A \subseteq FIN$ and $B \subseteq \mathbb{N}$ such that $FU(A) \in \mathcal{U}$, $FS(B) \in \mathcal{V}$ and there is a bijection $f : FU(A) \to FS(B)$ such that:*

(1) $f[A] = B$.

(2) For every non-empty $F \in [A]^{<\omega}$, $f\left(\bigcup_{a\in F} a\right) = \sum_{a\in F} f(a)$.

(3) $\beta f(\mathcal{U}) = \mathcal{V}$.

The following theorem shows that strongly summable ultrafilters and union ultrafilters are equivalent.

Theorem 5 ([2]) *Every strongly summable ultrafiter is additively isomorphic to a union ultrafilter. Every union ultrafilter is additively isomorphic to a strongly summable ultrafilter.*

3 Ultrafilters and forcing

It is well known that the partial order $([\mathbb{N}]^{\infty}, \subseteq^*)$, where $A \subseteq^* B$ means that $A \setminus B$ is finite (A is almost contained in B), is a forcing notion that adds a selective ultrafilter without adding subsets of $\mathbb{N}$.

We will consider an analogous forcing notion for ultrafilters on FIN.

The metric topology on FIN^{∞} is the topology generated by the sets of the form

$$[s] = \{X \in \mathrm{FIN}^{\infty} : s \sqsubset X\},$$

where $s \in \mathrm{FIN}^{<\infty}$ and $s \sqsubset X$ means that s is an initial segment of X.

Consider the binary relation defined on FIN^{∞} by $X \leq Y$ if X is a condensation of Y, that is, every element of X is a finite union of elements of Y, and so, $X \subseteq \mathrm{FU}(Y)$. We say that X is almost a condensation of Y, and write $X \leq^* Y$, to express that $X \subseteq^* \mathrm{FU}(Y)$.

Consider also the functions $r_n : \mathrm{FIN}^{\infty} \to \mathrm{FIN}^n$ given by setting $r_n(X)$ equal to the finite block sequence given by the first n elements of X.

Proposition 3 *The partial order $(FIN^{\infty}, \leq^*)$ is σ-closed.*

Proof. Let $\{A_n : n \in \omega\} \subseteq \mathrm{FIN}^{\infty}$ a sequence such that $A_{n+1} \leq^* A_n$ for every $n \in \omega$, i.e. $A_{n+1} \subseteq^* \mathrm{FU}(A_n)$. We have then that the family $\{\mathrm{FU}(A_n) : n \in \omega\}$ has the strong finite intersection property SFIP (every finite intersection of elements of the family is infinite). Let $a_0 \in \mathrm{FU}(A_0)$. If we have defined $a_0, a_1, ..., a_n$ take

$$a_{n+1} \in \bigcap_{i\leq n+1} \mathrm{FU}(A_i)$$

such that $\min(a_{n+1}) > \max(a_n)$. Let $A := \{a_n : n \in \omega\}$, then $A \subseteq^* \mathrm{FU}(A_n)$ for every $n \in \omega$ and therefore A is an almost condensation of each A_n. ■

Definition 6 *An ordered-union ultrafilter $\mathcal{U}$ on FIN is stable if for every sequence $\{D_n : n \in \omega\} \subseteq FIN^\infty$ such that $FU(D_n) \in \mathcal{U}$ for every $n \in \omega$, there is $E \in FIN^\infty$ such that $FU(E) \in \mathcal{U}$ and $E \leq^* D_n$ for every n.*

Theorem 6 ([4]) *Consider the forcing notion $(FIN^\infty, \leq^*)$ in V. Let G be a FIN^∞-generic filter over V. Then*

$$\mathcal{U}_G := \{A \subseteq FIN : \exists B \in G(FU(B) \subseteq A)\}$$

is a stable ordered-union ultrafilter in $V[G]$.

We include the proof for convenience.
Proof. Working in V, given $A \subseteq [\mathbb{N}]^{<\infty}$, consider the set

$$D_A := \{B \in \mathrm{FIN}^\infty : \mathrm{FU}(B) \subseteq A \text{ or } \mathrm{FU}(B) \cap A = \emptyset\}.$$

Note that A induces a partition $\mathrm{FU}(B) = (A \cap \mathrm{FU}(B)) \cup (([\mathbb{N}]^{<\infty} \setminus A) \cap \mathrm{FU}(B))$. By Hindman's theorem for unions, there is $C \in \mathrm{FIN}^\infty$ such that $\mathrm{FU}(C)$ is contained in one of the parts of the partition. Also, $C \subseteq \mathrm{FU}(C) \subseteq \mathrm{FU}(B)$ and therefore $C \leq B$. We conclude that D_A is dense, and so it contains an element of G. By Proposition 3 there are no new elements of FIN^∞ in $V[G]$, and thus $\mathcal{U}_G$ is an ultrafilter in $V[G]$. The stability of $\mathcal{U}_G$ also follows from Proposition 3, and that $\mathcal{U}_G$ is an ordered-union ultrafilter is clear. ■

Given a stable ordered-union ultrafilter $\mathcal{U}$, let $\mathcal{U}^\infty$ denote the set of sequences $A \in \mathrm{FIN}^\infty$ such that $\mathrm{FU}(A) \in \mathcal{U}$. Using the notation of the previous theorem, $\mathcal{U}_G^\infty = G$ and so $V[G] = V[\mathcal{U}_G]$.

Corollary 1 *If G is a FIN^∞-generic filter over V, then*

$$V[G] \models \text{“There is a strongly summable ultrafilter”}$$

4 Mathias style forcing

In [10], Mathias defined a forcing notion now known as *Mathias forcing*, given by

$$\mathbb{M} := \{(s, A) : s \in [\mathbb{N}]^{<\infty}, A \in [\mathbb{N}]^\infty \text{ and } \max(s) < \min(A)\}$$

with the order relation $(s, A) \leq (t, B)$ if and only of $t \subseteq s$, $A \subseteq B$ and $s \setminus t \subseteq B$.

This forcing notion was used to show that in Solovay's model where every set of real numbers is Lebesgue measurable [11], every subset of $[\mathbb{N}]^\infty$ is Ramsey.

If $G \subseteq \mathbb{M}$ is an $\mathbb{M}$-generic filter, then $\bigcup\{s \in \mathrm{FIN} : \exists A(s, A) \in G\}$ is called a Mathias real. Mathias forcing has the pure decision property and the hereditary genericity property. These properties are stated as follows.

Pure decision (or Prikry property): Given a sentence φ of the forcing language and $(s, A) \in \mathbb{M}$, there is $B \subseteq A$ such that (s, B) decides φ;

Hereditary genericity (or Mathias property): If x is a Mathias real, and $y \subseteq x$ is infinite, then y is also a Mathias real.

Various ways to adapt this forcing notion to the context of FIN^∞ have been studied by Eisworth, Matet, and García Ávila among others. We will concentrate here on the forcing notion called $\mathbb{P}_{\mathrm{FIN}}$ in [7] which is defined below.

Given $s \in \mathrm{FIN}^{<\infty}$, let $\overline{\max}(s) := \max(\{\max(a) : a \in s\})$, the maximum of the top block of s. Similarly, $\overline{\min}(s)$, for a finite or infinite block sequence s, denotes the minimal element of the first block of the sequence.

Let $A \in \mathrm{FIN}^\infty$ and $s \in \mathrm{FIN}^{<\infty}$,

$$A/s := \{a \in A : \overline{\max}(s) < \min(a)\}$$

the block sequence formed by the blocks of A that are above s.

Given $s \in \mathrm{FIN}^{<\infty}$ and a finite or infinite block sequence B, we say that s is an initial segment of B ($s \sqsubseteq B$) if $r_{|s|}(B) = s$.

Definition 7 *$\mathbb{P}_{FIN}$ is defined by*

$$\mathbb{P}_{FIN} := \{(s, A) : s \in FIN^{<\infty}, A \in FIN^\infty \text{ and } \overline{\max}(s) < \overline{\min}(A)\}$$

with the order relation $(s, A) \leq (t, B)$ if $t \sqsubseteq s$, $A \leq B$ and $s \setminus t \subseteq FU(B)$.

García Ávila in [7] proves that $\mathbb{P}_{FIN}$ has the pure decision property and asks if it also has the hereditary genericity property.

In this context, the hereditary genericity property is a natural variant of the Mathias property of $\mathbb{M}$ that can be stated replacing containment by condensation. Given a model V of ZFC, G a $\mathbb{P}_{FIN}$-generic filter over V, if a block sequence $\mathfrak{X}$ is $\mathbb{P}_{FIN}$-generic over V and $\mathfrak{Y} \leq \mathfrak{X}$ then $\mathfrak{Y}$ is also generic.

Here, a block sequence $\mathfrak{X} \in V[G]$ is $\mathbb{P}_{FIN}$-generic over V if there is a $\mathbb{P}_{FIN}$-generic filter H over V such that $\mathfrak{X} = \bigcup\{s : \exists A((s, A) \in H)\}$.

Whether this property holds was first asked in 2013 by García Ávila in her dissertation [6]. To answer this question we adapt Mathias' arguments from [10] and a theorem of Eisworth from [4] to the context of $\mathbb{P}_{FIN}$.

Definition 8 *Let $\mathcal{A} = \{A_s : s \in FIN^{<\infty}\} \subseteq FIN^{\infty}$. We say that $B \in FIN^{\infty}$ is a diagonalization of $\mathcal{A}$ if for every $s \in FIN^{<\infty}$ whose top block s is in $FU(B)$, $B/s \leq A_s$.*

The following theorem is due to Blass (see [1] 4.2). It establishes a Ramsey-like property equivalent to stability for ordered-union ultrafilters.

Given $X \subseteq [\mathbb{N}]^{<\infty}$, define

$$[X]^2_< := \{\{a, b\} \in [X]^2 : \max(a) < \min(b)\}.$$

Theorem 7 ([1]) *Let $\mathcal{U}$ be an ordered-union ultrafilter. Then $\mathcal{U}$ is stable if and only if for every $c : [[\mathbb{N}]^{<\infty}]^2_< \to 2$ there is $H \in \mathcal{U}$ such that $[H]^2_<$ is monochromatic.*

Blass' theorem actually establishes the equivalence of stability with several other properties, but we will use this particular one to prove a slight variation of a result of Eisworth (1.3 of [4]).

Corollary 2 *Let $\mathcal{U}$ be a stable ordered-union ultrafilter and $\mathcal{A} = \{A_b : b \in [\mathbb{N}]^{<\infty}\} \subseteq \mathcal{U}^{\infty}$. Then there is a diagonalization $B \in \mathcal{U}^{\infty}$, of $\mathcal{A}$.*

Proof. By the stability of $\mathcal{U}$, we can take $C \in \mathcal{U}^{\infty}$ such that $C \leq^* A_b$ for every $b \in [\mathbb{N}]^{<\infty}$ and a function $f : [\mathbb{N}]^{<\infty} \to \mathbb{N}$ such that $C/\{f(b)\} \leq A_b$. Define a coloring of $[\mathrm{FU}(C)]^2_<$ as follows:

$$g(\{a, b\}) = \begin{cases} 1 & f(a) < \min(b) \\ 0 & \text{otherwise} \end{cases}$$

By Theorem 7 there is $B \in \mathcal{U}^{\infty}$, a condensation of C such that $[\mathrm{FU}(B)]^2_<$ is monochromatic. Note that $c \restriction [\mathrm{FU}(B)]^2_< = 1$ since given $a \in \mathrm{FU}(B)$ we can find $b \in \mathrm{FU}(B)$ with $\min(b)$ arbitrarily big. Given $b \in \mathrm{FU}(B)$ we have then that $f(b) < \overline{\min}(B/b)$ and therefore $B/b \leq A_b$. ■

Corollary 2 also holds for families indexed by $\mathrm{FIN}^{<\infty}$. If $\{A_s : s \in \mathrm{FIN}^{<\infty}\} \subseteq \mathcal{U}^{\infty}$ we can find a diagonalization in $\mathcal{U}^{\infty}$, since given $b \in [\mathbb{N}]^{<\infty}$ we take A_b equal to some X such that $X \leq A_s$ for every s with top block b. Clearly, a diagonalization of $\{A_b : b \in [\mathbb{N}]^{<\infty}\}$ is a diagonalization for the initial family.

We follow now Mathias' construction in [10], adapted to block sequences.

Definition 9 *Let $(s, A) \in \mathbb{P}_{FIN}$ and $\mathcal{O} \subseteq FIN^{\infty}$. We say that (s, A) forces $\mathcal{O}$ and write $(s, A) \Vdash \mathcal{O}$ if*

$$[s, A] := \{X \in FIN^{\infty} : s \sqsubseteq X \text{ and } X/s \leq A\} \subseteq \mathcal{O}.$$

We say that (s, A) decides $\mathcal{O}$, and write $(s, A)||\mathcal{O}$, if (s, A) forces $\mathcal{O}$ or it forces $FIN^{\infty} \setminus O$.

Given a stable ordered-union ultrafilter $\mathcal{U}$, define the forcing notion

$$\mathbb{P}_{\mathcal{U}} := \{(s, A) : s \in \mathrm{FIN}^{<\infty} \text{ and } A \in \mathcal{U}^\infty\}$$

with the order inherited from $\mathbb{P}_{FIN}$.

Theorem 8 *Let $\mathcal{U}$ be a stable ordered-union ultrafilter on FIN. Let $\mathcal{O} \subseteq FIN^\infty$ be open with respect to the product topology. Then, there is $B \in \mathcal{U}^\infty$ such that $(\emptyset, B)||\mathcal{O}$.*

Proof. Let $N = \{\{n\} : n \in \mathbb{N}\}$. For every $s \in \mathrm{FIN}^{<\infty}$, choose $A_s \in \mathcal{U}^\infty$ such that $(s, A_s)||\mathcal{O}$ if possible, otherwise put $A_s = N/s$. Let A be a diagonalization of $\{A_s : s \in \mathrm{FIN}^{<\infty}\}$. Define $\phi : \mathrm{FIN}^{<\infty} \to 3$ by

$$\phi(s) = \begin{cases} 0 & \text{if } (s, A_s) \Vdash \mathcal{O} \\ 1 & \text{if } (s, A_s) \Vdash \mathrm{FIN}^\infty \setminus \mathcal{O} \\ 2 & \text{otherwise.} \end{cases}$$

If $\phi(s) = 2$ then $\{b \in \mathrm{FU}(A/s) : \phi(s \cup \{b\}) = 2\} \in \mathcal{U}$. To show this, put, for every $i = 0, 1, 2$, $S_i = \{b \in \mathrm{FU}(A/s) : \phi(s \cup \{b\}) = i\}$. Then, if $b \in S_0$, $(s \cup \{b\}, A/(s \cup \{b\})) \Vdash \mathcal{O}$.

Notice that $\{B \in \mathrm{FIN}^\infty : s \sqsubset B \text{ and } B/s \subseteq S_0\} =$

$$= \bigcup_{b \in S_0} \{B \in \mathrm{FIN}^\infty : s \cup \{b\} \sqsubset B \text{ and } B/(s \cup \{b\}) \subseteq S_0/(s \cup \{b\})\}.$$

This union is contained in $\mathcal{O}$, since for every $b \in S_0$, $(s \cup \{b\}, A/(s \cup \{b\})) \Vdash \mathcal{O}$.

Similarly, if $b \in S_1$, $\{B \in \mathrm{FIN}^\infty : s \sqsubset B \text{ and } B/s \subseteq S_1\}$ is disjoint from $\mathcal{O}$.

If $\phi(s) = 2$, then (s, A_s) does not decide $\mathcal{O}$, thus when defining A_s, this set was set $A_s = N/s$, since for no $C \in \mathcal{U}^\infty$, $(s, C)||\mathcal{O}$. Thus, $S_0 \notin \mathcal{U}$, because if $S_0 \in \mathcal{U}$ then $A_s \in \mathcal{U}^\infty$ and $(s, A/s) \Vdash \mathcal{O}$, but there is no $C \in \mathcal{U}^\infty$ with this property. Similarly, $S_1 \notin \mathcal{U}$. Therefore $S_2 \in \mathcal{U}$ as $S_0 \cup S_1 \cup S_2 = \mathrm{FU}(A/s)$.

Suppose now that $\phi(\emptyset) = 2$. For every s, put $B_s = \mathrm{FU}(A/s)$ if $s \not\subseteq A$ or $\phi(s) \neq 2$, and $B_s = \{b \in \mathrm{FU}(A/s) : \phi(s \cup b) = 2\}$ if $\phi(s) = 2$.

Since $\mathcal{U}$ is an ordered-union ultrafilter, we can take for each $s \in \mathrm{FIN}^{<\infty}$, $C_s \in \mathcal{U}^\infty$ such that $\mathrm{FU}(C_s) \subseteq B_s$. Let D be a diagonalization of $\{C_s : s \in \mathrm{FIN}^{<\infty}\}$.

Let us show that for $s \subseteq D$ finite, $\phi(s) = 2$. Let $s \subseteq D$ finite a counterexample of minimal cardinality. By our assumption $s \neq \emptyset$. Let b be the top block of s; and let $t = s \setminus b$. We have then that $\phi(t) = 2$, and since $b \in D/t \subseteq \mathrm{FU}(C_t) \subseteq B_t$, we get that $\phi(t \cup \{b\}) = \phi(s) = 2$ which contradicts our choice of s.

This leads us to a contradiction, because if $(\emptyset, D)$ does not decide $\mathcal{O}$, then there is $E \leq D$ such that $E \in \mathcal{O}$, and thus there is n such that $(r_n(E), N/r_n(E)) \subseteq \mathcal{O}$ and therefore $\phi(r_n(E)) = 0$ contradicting what we have just showed. Then $\phi(\emptyset)$ cannot be 2. ■

Definition 10 *Let $\mathcal{U}$ a stable ordered-union ultrafilter, $D \subseteq \mathbb{P}_{\mathcal{U}}$ dense open and $s \in FIN^{<\infty}$. We say that $X \in FIN^{\infty}$ captures (s, D) if*

(i) $X \in \mathcal{U}^{\infty}$,

(ii) X *is above* s, *i.e.* $\overline{max}(s) < \overline{min}(X)$,

(iii) For every $Y \leq X$ *there is* $t \sqsubset Y$ *such that* $(s \cup t, X/t) \in D$.

Lemma 1 *Let $\mathcal{U}$ a stable ordered-union ultrafilter and $D \subseteq \mathbb{P}_{\mathcal{U}}$ dense open. For every $s \in FIN^{<\infty}$, there exists $X \in \mathcal{U}^{\infty}$ such that X captures (s, D)*

Proof. Let $N = \{\{n\} : n \in \mathbb{N}\}$. Fix $s \in \mathrm{FIN}^{<\infty}$, and let $C = N/s$. For every $t \in \mathrm{FIN}^{<\infty}$, if $\max(s) < \min(t)$, let $A_t \in \mathcal{U}^{\infty}$ be such that $(s \cup t, A_t) \in D$ if such an A_t exists, otherwise or if t is not above s, put $A_t = C/t$. Let $B \in \mathcal{U}^{\infty}$ be a diagonalization of the collection $\{A_t : t \in \mathrm{FIN}^{<\infty}\}$. For every t of which the top block is in $\mathrm{FU}(B)$, if there is some $B' \in \mathcal{U}^{\infty}$ such that $(s \cup t, B') \in D$, then $(s \cup t, B/t) \in D$.

Let
$$\mathcal{O} = \{A \in \mathrm{FIN}^{\infty} : A \leq B \to \exists t \sqsubset A(s \cup t, B/t) \in D\}.$$

The set $\mathcal{O}$ is open in the product topology of FIN^{∞}, so there is $E \in \mathcal{U}^{\infty}$ such that $(\emptyset, E)||\mathcal{O}$.

Since $\mathcal{U}$ is an ordered-union ultrafilter, there is $X \in \mathcal{U}^{\infty}$ such that $\mathrm{FU}(X) \subseteq \mathrm{FU}(B) \cap \mathrm{FU}(E) \in \mathcal{U}$; and since D is dense, there are $u \sqsubset X$ and $X' \leq X$ such that $(s \cup u, X') \in D$ (and obviously $(s \cup u, X') \leq (s, X)$).Then, $(s \cup u, B/u) \in D$, and thus $u \cup X' \in \mathcal{O}$. Since $u \cup X' \subseteq \mathrm{FU}(X) \subseteq \mathrm{FU}(E)$, $u \cup X' \leq E$, and therefore, $(\emptyset, E) \Vdash \mathcal{O}$.

To verify that X captures (s, D), take $Y \leq X$ There is $t \sqsubset Y$ such that $(s \cup t, B/t) \in D$, and as D is open, $(s \cup t, X/t) \in D$.

■

Theorem 9 *Let $\mathcal{U} \in V$ be a stable ordered-union ultrafilter and $\mathcal{X} \in FIN^{\infty}$. Then $\mathcal{X}$ is $\mathbb{P}_{\mathcal{U}}$-generic over V if and only if $\mathcal{X} \leq^* A$ for every $A \in \mathcal{U}^{\infty}$.*

Proof. Suppose first that $\mathcal{X}$ is $\mathbb{P}_{\mathcal{U}}$-generic over V. For every $X \in \mathcal{U}^{\infty}$, the set $\{(s, A) \in \mathbb{P}_{\mathcal{U}} : A \leq X\}$ is dense, and therefore there is $(s, A) \in \mathbb{P}_{\mathcal{U}}$ such that $s \sqsubset \mathcal{X}$ and $\mathcal{X}/s \leq X$. We have then that the generic filter associated to $\mathcal{X}$ has non-empty intersection with this dense set. In other words, there is some $(s, A) \in \mathbb{P}_{\mathcal{U}}$ such that $s \sqsubset \mathcal{X}$ and $\mathcal{X}/s \leq A \leq X$. Thus $\mathcal{X} \leq^* X$.

Suppose now that $\mathcal{X} \leq^* X$ for every $X \in \mathcal{U}^{\infty}$, and let $D \in V$ dense open subset of $\mathbb{P}_{\mathcal{U}}$. Working in V, by Lemma 1 let, for every $s \in \mathrm{FIN}^{<\infty}$, $X_s \in \mathcal{U}^{\infty}$ be such that X_s captures (s, D) and let $X \in \mathcal{U}^{\infty}$ be a diagonalization of $\{X_s : s \in \mathrm{FIN}^{<\infty}\}$. By our assumption, $\mathcal{X} \leq^* X$. Let $m \in \mathbb{N}$ be such that $\mathcal{X}/r_m(\mathcal{X}) \leq X$. The top block of

$r_{m+1}(\mathcal{X})$ is in FU(X), so $\mathcal{X}/r_{m+1}(\mathcal{X}) \leq X/r_{m+1}(\mathcal{X}) \leq X_{r_{m+1}(X)}$ and thus $\mathcal{X}/r_{m+1}(\mathcal{X})$ captures $(r_{m+1}(\mathcal{X}), D)$.

Thus in V,

$$(*)\ \forall Y \leq X/r_{m+1}(\mathcal{X}) \exists t \sqsubset Y \text{ such that } (r_{m+1}(\mathcal{X}) \cup t, X/t) \in D.$$

Let $P = \{t \in \mathrm{FIN}^{<\infty} : t \sqsubset X/r_{m+1}(\mathcal{X}) \text{ and } (r_{m+1}(\mathcal{X}) \cup t, X/t) \notin D\}$. Define a partial order on P by $s \leq t$ if $t \sqsubseteq s$.

We have then that $(P, \leq) \in V$, and that property $(*)$ is equivalent to the assertion that $(P, \leq)$ is well founded. Since to be well founed is absolute, there is in the real world $t \sqsubset X/r_{m+1}(\mathcal{X})$ such that $(r_{m+1}(\mathcal{X}) \cup t, X/t) \in D$. Also, $r_{m+1}(\mathcal{X}) \cup t \sqsubset \mathcal{X}$, and $\mathcal{X}/t \leq X/t$. Thus, $\mathcal{X}$ is generic over V. ■

Lemma 2 *1. Let $\mathcal{X}$ be $\mathbb{P}_{FIN}$-generic over V, and let $G = \{A \in FIN^\infty \cap V : \mathcal{X} \leq^* A\}$. Then, G is a FIN^∞-generic fiter over V and $\mathcal{X}$ is $\mathbb{P}_{\mathcal{U}_G}$-generic over $V[G]$.*

2. Let G be a FIN^∞-generic filter over V and $\mathcal{X}$ $\mathbb{P}_{\mathcal{U}_G}$-generic over $V[G]$. Then $\mathcal{X}$ is $\mathbb{P}_{FIN}$-generic over V.

Proof.

1. To show that G is FIN$^\infty$-generic over V, let $D \subseteq \mathrm{FIN}^\infty$ be dense open $D \in V$and let $D' = \{(s, A) : A \in D\}$. We have then that D' is dense and thus there is $(s, A) \in D'$ such that $s \sqsubset \mathcal{X}$ and $\mathcal{X}/s \leq A$, giving that $A \in G$, and thus G is FIN$^\infty$-generic over V. Since FIN$^\infty$ is a σ-closed partial order

   we have that $\mathfrak{X}$ is $\mathbb{P}_{\mathcal{U}_G}$-generic over $V[G]$.

(2) Let $\mathbb{P}_{FIN} \supseteq D \in V$ dense open, and put

$$D' := \{(s, A) \in D : A \in \mathcal{U}\}$$

$$D'' := \{A \in \mathrm{FIN}^\infty : \exists s((s, A) \in D)\}.$$

Note that $D'' \in V$ is dense open, and thus, since G is FIN$^\infty$-generic, $D' \in V[G]$ is also dense open. Now, as $\mathfrak{X}$ is generic over $V[G]$ there is $(s, A) \in D' \subseteq D$ such that $s \sqsubseteq \mathfrak{X}$ and $\mathfrak{X}/s \leq A$. We have then that $\mathfrak{X}$ is $\mathbb{P}_{FIN}$-generic over V.

■

This shows that a forcing extension obtained using $\mathbb{P}_{FIN}$ can be seen as an iteration, first adding a stable ordered-union ultrafilter $\mathcal{U}$ by $(\mathrm{FIN}^\infty, \leq^*)$ and then forcing with $\mathbb{P}_{\mathcal{U}}$.

Corollary 3 *Let $\mathfrak{X}$ a $\mathbb{P}_{FIN}$-generic block sequence over V and $\mathfrak{Y} \leq \mathfrak{X}$. Then $\mathfrak{Y}$ is also generic over V.*

Proof. It follows from Theorem 9 and Lemma 2. ■

5 Concluding remarks

In [4], Eisworth defines a class of subsets of FIN^{∞} which he calls Matet-adequate families.

Definition 11 *A family $\mathcal{H} \subseteq FIN^{\infty}$ is said to be Matet-adequate if:*

(1) For every $A \in \mathcal{H}$ and $B \in FIN^{\infty}$, if $A =^* B$ then $B \in \mathcal{H}$.

(2) For every $A \in \mathcal{H}$ and $A \leq^* B$, $B \in \mathcal{H}$.

(3) For every family $\{A_n : n \in \omega\} \subseteq \mathcal{H}$ such that $A_{n+1} \leq^* A_n$ for every $n \in \omega$, there exists $B \in \mathcal{H}$ such that $B \leq^* A_n$ for every $n \in \omega$.

(4) For every $A \in \mathcal{H}$ and every coloring $c : FU(A) \to 2$ there exists $B \leq A$ in $\mathcal{H}$ such that $FU(B)$ is monochromatic.

If $\mathcal{U}$ is a stable ordered-union ultrafilter, then $\mathcal{U}^{\infty}$ is a Matet-adequate family. Eisworth shows that if $\mathcal{H}$ is a Matet-adequate family, forcing with $(\mathcal{H}, \leq^*)$ adds a stable ordered-union ultrafilter.

Given a Matet-adequate family $\mathcal{H}$, the forcing $\mathbb{P}_{FIN}$ can be modified to define $\mathbb{P}_{\mathcal{H}}$ as he collection of conditions (s, A) where $s \in \mathrm{FIN}^{<\infty}$ and $A \in \mathcal{H}$, with the same order relation as before.

Standard arguments give that a generic $\mathbb{P}_{\mathcal{H}}$ extension can be obtained first adding a stable ordered-union ultrafilter $\mathcal{U}$ with $(\mathcal{H}, \leq^*)$ and then forcing with $\mathbb{P}_{\mathcal{U}}$. From this we can argue as in [7], [4] and as in the previous section to show that $\mathbb{P}_{\mathcal{H}}$ has both the pure decision property and the hereditary genericity property.

In a forthcoming article ([3]) several of these results are generalized to the context of topological Ramsey spaces of [12].

References

[1] A. Blass. Ultrafiters related to Hindman's finite-unions theorem and its extensions. *Contemporary Mathematics* 65, pp. 89–124. AMS, Providence, 1987.

[2] A. Blass and N. Hindman. On strongly summable ultrafilters and union ultrafilters. *Transactions of the American Mathematical Society* 304(1):83–99, 1987.

[3] C. A. Di Prisco, J. G. Mijares and J. Nieto. Local Ramsey Theory. An abstract approach. *Mathematical Logic Quarterly*, to appear.

[4] T. Eisworth. Forcing and ordered-union ultrafilters. *The Journal of Symbolic Logic* 67(1):449–464, 2002.

[5] E. Ellentuck. A new proof that analytic sets are Ramsey. *The Journal of Symbolic Logic* 39(1):163–165, 1974.

[6] L. M. García Ávila. *Forcing arguments in infinite Ramsey theory.* Doctoral Dissertation. Universitat de Barcelona, 2013.

[7] L. M. García Ávila. A forcing notion related to Hindman's theorem. *Archive for Mathematical Logic* 54(1-2):133–159, 2015.

[8] N. Hindman. Finite sums for sequences within cells of a partition of $\mathbb{N}$. *Journal of Combinatorial Theory* (A) 17(1):1–11, 1974.

[9] P. Matet. Some filters of partitions. *The Journal of Symbolic Logic* 53(2):540–553, 1988.

[10] A. R. D. Mathias. Happy Families. *Annals of Mathematical Logic* 12(1):59–111, 1977.

[11] R. M. Solovay. A model of set-theory in which every set of reals is Lebesgue measurable. *Annals of Mathematics* 92(1):1–56, 1970.

[12] S. Todorcevic. *Introduction to Ramsey spaces.* Princeton University Press, Princeton, 2010.

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