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An Algebraic Approach to Orthologic

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Abstract

The literature about quantum theories emphasizes that the algebraic structures associated to orthodox quantum mechanics are non-distributive. In this paper we present a usual development on quantum algebras, the ortholattices, and a correspondent deductive system associated to them, the orthologic. Then, we show the adequacy between the algebraic ortholattices and the propositional orthologic using specifically algebraic models.

Keywords: Ortholattice, orthologic, algebraic model.

Introduction

The algebras of quantum theories are non-distributive and the corresponding logics are also non-distributive relative to the operators of conjunction and disjunction. Because of that it is not possible to use the famous theorem of Stones Isomorphism to establish their completeness. For more details on the beginning of quantum theories related to logic we suggest [1], [3], [6], [8] and [19].

Considering this non-distributivity originated from the non-distributivity of closed Hilbert spaces used in the foundation of Physical Theories, there is a tradition to associate to the quantum theories the basic algebraic structure named ortholattice as a first algebraic approximation.

Goldblatt [9] and Dalla Chiara, Giuntini and Greechie [6] have used an interesting semantic in Kripke's style to connect the algebraic models of ortholattices with the propositional quantum logic, the orthologic.

In this paper, we present algebraic aspects of quantum algebras and, then, we introduce a short deductive system very similar to those presented in above papers. We show some derivations on this Tarski system.

As an original contribution, we present a completely algebraic proof of soundness and completeness of orthologic relative to the ortholattices.

1 Algebras of quantum theories

Here we just present some elements of algebraic logic for the development of quantum logics. These elements are well known and can be met in several texts as [17], [18], [14], [5], [7] and [12].

Definition 1.1 *(Lattice) A lattice is an algebraic structure* $\mathbb{L} = \langle L, \curlywedge, \curlyvee \rangle$ *such that* L *is a non-empty set,* $\curlywedge$ *and* $\curlyvee$ *are two binary operations on* L *and for all* $a, b, c \in L$*:*

L_1 $(a \curlywedge b) \curlywedge c = a \curlywedge (b \curlywedge c)$ *and* $(a \curlyvee b) \curlyvee c = a \curlyvee (b \curlyvee c)$ *[associativity]*

L_2 $a \curlywedge b = b \curlywedge a$ *and* $a \curlyvee b = b \curlyvee a$ *[commutativity]*

L_3 $(a \curlywedge b) \curlyvee b = b$ *and* $(a \curlyvee b) \curlywedge b = b$ *[absorption].*

Proposition 1.2 *If* $\mathbb{L} = \langle L, \curlywedge, \curlyvee \rangle$ *is a lattice and* $a, b \in L$*, then it holds:*

L_4 $a \curlywedge a = a$ *and* $a \curlyvee a = a$ [idempotency]

L_5 $a \curlywedge b = a \Leftrightarrow a \curlyvee b = b$ [ordering].

Using condition L_5, we can define a relation of partial order on $\mathbb{L} = \langle L, \curlywedge, \curlyvee \rangle$.

Definition 1.3 *(Order)* $a \leq b \Leftrightarrow a \curlywedge b = a \Leftrightarrow a \curlyvee b = b$.

Proposition 1.4 *If* $\mathbb{L} = \langle L, \curlywedge, \curlyvee \rangle$ *is a lattice and* $a, b, c, d \in L$*, then:*

L_6 $a \leq a \curlyvee b$ *and* $b \leq a \curlyvee b$

L_7 $a \curlywedge b \leq a$ *and* $a \curlywedge b \leq b$

L_8 $a \leq c$ *and* $b \leq c \Rightarrow a \curlyvee b \leq c$

L_9 $c \leq a$ *and* $c \leq b \Rightarrow c \leq a \curlywedge b$

L_{10} $a \leq c$ *and* $b \leq d \Rightarrow a \curlyvee b \leq c \curlyvee d$

L_{11} $a \leq c$ *and* $b \leq d \Rightarrow a \curlywedge b \leq c \curlywedge d$.

We have defined lattice as an algebraic structure, but this concept can also be introduced as an ordering structure $\mathbb{L} = \langle L, \leq \rangle$.

Definition 1.5 *(Partial order) A binary relation* $\leq$ *on a non-empty set* L *is a partial order if the relation* $\leq$ *is reflexive, antisymmetric and transitive.*

Definition 1.6 *(Poset) A partially ordered set is a pair* $\langle L, \leq \rangle$ *such that* L *is a non-empty set and* $\leq$ *is a partial order on* L*.*

Definition 1.7 *(Supremum) Let* $\langle L, \leq \rangle$ *be a poset and* $a, b \in L$*. A supremum of* $\{a, b\}$*, if it exists, is an element* $c \in L$ *such that:*

(i) $a \leq c$ *and* $b \leq c$

(ii) *if* $a \leq d$ *and* $b \leq d$*, then* $c \leq d$*.*

A supremum, if it exists, is unique.

An infimum of $\{a, b\}$ is defined dually. It is unique, if it exists.

It is usual to denote the supremum of $\{a, b\}$ by $\sup\{a, b\}$ or $a \curlyvee b$ and the infimum of $\{a, b\}$ by $\inf\{a, b\}$ or $a \curlywedge b$. The supremum of $\{a, b\}$ is also named the *least upper bound* of $\{a, b\}$ and the infimum of $\{a, b\}$ is called the *greatest lower bound* of $\{a, b\}$.

If $\langle L, \leq \rangle$ is a poset such that for all $a, b \in L$ there exist $\inf\{a, b\}$ and $\sup\{a, b\}$, then the algebraic structure determined by $\langle L, \curlywedge, \curlyvee \rangle$ in which:

$$a \curlywedge b = \inf\{a, b\} \qquad \text{and} \qquad a \curlyvee b = \sup\{a, b\}$$

is a lattice.

It is straightforward to observe that these operations $\curlywedge$ and $\curlyvee$ satisfy the associative, commutative and absorption properties.

We can easily prove that the laws L_1 to L_{11} hold for the poset $\langle L, \leq \rangle$.

This way we can always see a lattice as a structure $\mathbb{L} = \langle L, \leq, \curlywedge, \curlyvee \rangle$.

Lemma 1.8 *If* $\mathbb{L} = \langle L, \leq, \curlywedge, \curlyvee \rangle$ *is a lattice, then:*

L_{12} $(a \curlywedge b) \curlyvee (a \curlywedge c) \leq a \curlywedge (b \vee c)$

L_{13} $a \curlyvee (b \curlywedge c) \leq (a \curlyvee b) \curlywedge (a \curlyvee c)$.

Proof. The result follows from L_6, L_7, and L_8. ■

Definition 1.9 *(Distributive lattice) A lattice* $\mathbb{L} = \langle L, \leq, \curlywedge, \curlyvee \rangle$ *is distributive if the following distributive laws are valid for all* $a, b, c \in L$*:*

L_{14} $(a \curlywedge b) \curlyvee c = (a \curlyvee c) \curlywedge (b \curlyvee c)$ *and* $(a \curlyvee b) \curlywedge c = (a \curlywedge c) \curlyvee (b \curlywedge c)$.

These are the right distributive laws and, due to the commutative property, the left distributive laws are also valid. Besides, only one of these two distributive laws would be enough to characterize the distributive property [14].

Definition 1.10 *(Lattices with* 0 *and* 1*) Let* $\mathbb{L} = \langle L, \leq, \curlywedge, \curlyvee \rangle$ *be a lattice. If* $\mathbb{L}$ *has the least element with respect to the order* $\leq$*, then this element is called the zero of* $\mathbb{L}$ *and is denoted by* 0*. On the other hand, if the lattice* $\mathbb{L}$ *has the greatest element with respect to the order* $\leq$*, then this element is called the one of* $\mathbb{L}$ *and it is denoted by* 1*.*

If the lattice $\mathbb{L}$ has the elements 0 and 1, then for every $a \in L$:

L_{15} $a \curlywedge 0 = 0$ and $a \curlyvee 0 = a$

L_{16} $a \curlywedge 1 = a$ and $a \curlyvee 1 = 1$.

We denote a lattice with 0 and 1 by $\mathbb{L} = \langle L, \leq, \curlywedge, \curlyvee, 0, 1\rangle$.

Definition 1.11 *(Pseudo-complement) Let* $\mathbb{L} = \langle L, \leq, \curlywedge, \curlyvee, 0, 1\rangle$ *be a lattice with* 0 *and* 1*. For* $a \in L$*, if there exists the element* $-a = \max\{y \in L : a \curlywedge y = 0\}$ *in* L*, then* $-a$ *is called the pseudo-complement of* a*.*

Definition 1.12 *(Pseudo-complemented lattice) A lattice* $\mathbb{L}$ *is called pseudo-complemented if every element* $a \in L$ *has a pseudo-complement* $-a \in L$*.*

Definition 1.13 *(Complement) Let* $\mathbb{L} = \langle L, \leq, \curlywedge, \curlyvee, 0, 1\rangle$ *be a lattice with* 0 *and* 1*. If* $a \in L$*, then an element* $\overline{a} \in L$ *is called a complement of* a *in* $\mathbb{L}$ *if:*

L_{17} $a \curlywedge \overline{a} = 0$

L_{18} $a \curlyvee \overline{a} = 1$.

The complement $\overline{a}$ is a pseudo-complement. But, for example, the intuitionistic pseudo-complement of Intuitionistic Logic is not a complement.

Definition 1.14 *(Complemented lattice) The lattice* $\mathbb{L} = \langle L, \leq, \curlywedge, \curlyvee, 0, 1\rangle$ *is called complemented if every element in* L *has a complement in* L*. If every element of* L *has exactly one complement, then the lattice* $\mathbb{L}$ *is called uniquely complemented.*

If the complement of a is unique, we will denote it by $\sim a$. If a lattice $\mathbb{L}$ is uniquely complemented, then we write $\mathbb{L} = \langle L, \leq, \sim, \curlywedge, \curlyvee, 0, 1\rangle$.

Lemma 1.15 *Let* $\mathbb{L} = \langle L, \leq, \curlywedge, \curlyvee, 0, 1\rangle$ *be a distributive lattice with* 0 *and* 1*. If there exists a complement of* a*, then it is unique.*

Proof. If y and z are two complements of a, then $a \curlywedge y = 0$, $a \curlyvee y = 1$, $a \curlywedge z = 0$, and $a \curlyvee z = 1$. As $z = 0 \curlyvee z = (a \curlywedge y) \curlyvee z = (a \curlyvee z) \curlywedge (y \curlyvee z) = 1 \curlywedge (y \curlyvee z) = y \curlyvee z$, we have, $y \leq z$. Analogously, $z \leq y$ and, hence, $z = y$. ■

Definition 1.16 *(Boolean algebra) A Boolean algebra* $\mathcal{B}$ *is a distributive and complemented lattice.*

The next results are particular cases of quantum algebras and good references are the texts [6] and [15].

Definition 1.17 *(Poset with involution) Let* $\mathbb{L} = \langle L, \leq, 0, 1\rangle$ *be a poset. An involution on* $\mathbb{L}$ *is a unary operation, denoted by* $'$*, such that for all* $a, b \in L$*:*

L_{19} $a = a''$

L_{20} $a \leq b \Rightarrow b' \leq a'$.

Then, $\mathbb{L} = \langle L, \ ', \leq, 0, 1\rangle$ is a poset with involution.

Proposition 1.18 *If* $\mathbb{L} = \langle L, \ ', \leq, 0, 1\rangle$ *is a poset with involution, then the De Morgan's laws hold:*

L_{21} $\quad (a \curlywedge b)' = a\,' \curlyvee b\,'$

L_{22} $\quad (a \curlyvee b)' = a\,' \curlywedge b\,'$.

Indeed, in view of L_{20} in $\mathbb{L} = \langle L, \leq, \ ', 0, 1\rangle$ the conditions L_{20}, L_{21} and L_{22} are equivalent.

Besides, in this case, $\sup\{a, b\}$ is defined if, and only if, $\inf\{a, b\}$ is also defined.

Definition 1.19 *(Ortholattice) An ortholattice is a complemented lattice with involution.*

We denote a such structure by $\mathbb{L} = \langle L, \leq, \ ', \curlywedge, \curlyvee, 0, 1\rangle$.

So in an ortholattice all the conditions L_1 - L_{22}, except distributivity L_{14}, are valid.

This way of including properties mirrors the achievement of Boolean algebra as in the tradition of Heyting algebras with intermediate algebras (Heyting algebra - Boolean algebra), with the difference of non-distributivity. The way from any ortholattice to a Boolean algebra has so many points and we can add several additional conditions or algebraic axioms depending on the path.

Following this context, the ortholattices are considered basic quantum structures.

In this paper we concentrate on ortholattices using only algebraic approach, which we shall posteriorly apply to the other quantum algebraic systems.

Like a last structure, let's define Kripke models as [6].

Definition 1.20 *(Kripke model) A model in the Kripke style for a language* $\boldsymbol{L}$ *has the following form:* $\mathcal{K} = (W, \vec{R}, \vec{o}, \mathcal{P}(W), v)$*, such that:*

(i) W *is a non-empty set of possible worlds;*

(ii) $\vec{R}$ *is a sequence of relations over* W*;*

(iii) $\vec{o}$ *is a sequence of operations defined over* W*;*

(iv) *the subsystem* $(W, \vec{R}, \vec{o})$ *is called the frame of* $\mathcal{K}$*;*

(v) $\mathcal{P}(W)$ *is the set of all subsets of* W*;*

(vi) $v : Var(\boldsymbol{L}) \to \mathcal{P}(W)$ *is a valuation that applies each variable into the set of all worlds where the variable is true or valid;*

(vii) *each valuation must preserve conditions that depend on the operators* $\vec{o}$ *of* $\boldsymbol{L}$*;*

(viii) *the valuations must be extended for the set of all formulas of* $\boldsymbol{L}$*.*

Usually we have only one binary relation R in the sequence $\vec{R}$, called the accessibility relation.

Considering that we almost always have relations involved in Kripke models, they are not exactly algebraic models, but a combination of algebraic and relational structures.

2 Logic of ortholattices

We present in this section the Orthologic, denoted by $\mathcal{OL}$, the logic of ortholattices, in a similar version to [9] and [6] and oriented by [20].

There is an interesting tradition on logic for quantum theories. We mention the following references: [10], [2], [11], [4] and [15].

The orthologic formalizes, in the logical language, some of essential characteristics of quantum theories that are unveiled by the orthoalgebras.

We do not have an algebraic conditional operator and circumvent this situation using a deductive system without any logical implication. We found this strategy for the first in [9].

The language of $\mathcal{OL}$ is indicated by $\mathbf{L}$.

The above literature shows aspects of quantum logics.

The propositional language $\mathbf{L}$ has exactly the operators $\neg$ for negation, and $\wedge$ for conjunction. Thus we take $\mathbf{L} = \{\neg, \wedge\}$.

The set of formulas of $\mathcal{OL}$ is denoted by $For(\mathbf{L})$ and the set of propositional variables by $Var(\mathbf{L}) = \{p_1, p_2, p_3, \ldots\}$. Of course $Var(\mathbf{L}) \subseteq For(\mathbf{L})$.

Thus, $For(\mathbf{L})$ is constructed from $Var(\mathbf{L})$ using only the symbols in $\mathbf{L} = \{\neg, \wedge\}$.

We do not have the disjunction $\vee$ as a basic operator in the language $\mathbf{L}$, but considering that in any ortholattice the De Morgan laws hold, we can define the disjunction of $\mathbf{L}$ by:

$$\varphi \vee \psi =_{df} \neg(\neg\varphi \wedge \neg\psi).$$

Definition 2.1 *(Configuration) For $\Sigma \cup \{\psi\} \subseteq For(\boldsymbol{L})$, a configuration is an expression of type $\Sigma \vdash \psi$.*

These configurations are schemes of formulas and we mean that we derive the consequence at right of $\vdash$ from the antecedent (a set of premises) at left of $\vdash$. The antecedent is a set of formulas and it is not required that it be a sequence or a finite multiset as in some calculus of sequents.

Derivation is a figure composed by a sequence of configurations.

For a formal definition we need to explicit the rules for derivations.

In general, if for $i \in \{1, 2, \ldots, n\}$, $\Sigma_i \cup \{\psi_i\} \subseteq For(\mathbf{L})$, then each rule has the form:

$$\frac{\Sigma_1 \vdash \psi_1, \ldots, \Sigma_{n-1} \vdash \psi_{n-1}}{\Sigma_n \vdash \psi_n},$$

with the meaning that from the premises, the configurations above the line, each rule permits the deduction of configuration $\Sigma_n \vdash \psi_n$.

The rules without premises are special cases, where the set of premises is empty, such that instead of: $\dfrac{\emptyset}{\Sigma \vdash \psi}$ we just write $\Sigma \vdash \psi$.
Of course, the configuration $\vdash \varphi$ must be understood as $\emptyset \vdash \varphi$.

Now, we present the properties of derivability for the logical system $\mathcal{OL}$.

This system does not have axioms, but only rules determined by the following configurations.

Rules without premises:

(ROL_1) $\{\varphi\} \vdash \varphi$ (auto-deductibility)

(ROL_2) $\{\varphi\} \vdash \neg\neg\varphi$ (double negation)

(ROL_3) $\{\neg\neg\varphi\} \vdash \varphi$ (double negation)

(ROL_4) $\{\varphi \wedge \psi\} \vdash \varphi$ (simplification)

(ROL_5) $\{\varphi \wedge \psi\} \vdash \psi$ (simplification)

(ROL_6) $\{\varphi \wedge \neg\varphi\} \vdash \sigma$ (explosion)

Rules with one premise:

(ROL_7) $\dfrac{\Gamma \vdash \varphi}{\Gamma \cup \Sigma \vdash \varphi}$ (monotonicity)

(ROL_8) $\dfrac{\{\psi\} \vdash \varphi}{\{\neg\varphi\} \vdash \neg\psi}$ (contraposition)

(ROL_9) $\dfrac{\{\varphi, \psi\} \vdash \sigma}{\{\varphi \wedge \psi\} \vdash \sigma}$ (left conjunction)

Rules with two premises:

(ROL_{10}) $$\frac{\Gamma \vdash \varphi,\ \Delta \cup \{\varphi\} \vdash \psi}{\Gamma \cup \Delta \vdash \psi} \text{ (cut)}$$

(ROL_{11}) $$\frac{\{\psi\} \vdash \varphi,\ \{\psi\} \vdash \neg\varphi}{\vdash \neg\psi} \text{ (absurdity)}$$

(ROL_{12}) $$\frac{\Gamma \vdash \varphi,\ \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi} \text{ (right conjunction).}$$

From auto-deductibility, monotonicity and cut, we observe that $\mathcal{OL}$ is a logic of Tarski. These three rules are considered structural rules, that is, rules without connectives. The other rules have the aim to put the particularities of an ortholattice in the propositional context.

Definition 2.2 *(Derivation) A derivation in $\mathcal{OL}$ is a finite sequence of configurations $\Sigma \vdash \psi$ such that each element in the sequence is a premise, or a rule without premises, or a conclusion of a rule whose premises are previous elements in the sequence.*

Definition 2.3 *(Derivable formula) A formula ψ is derivable from Σ if there is a derivation such that the last element of derivation is the configuration $\Sigma \vdash \psi$.*

Definition 2.4 *(Theorem) A formula ψ is a theorem of $\mathcal{OL}$ if it is derivable from the empty set, that is, $\emptyset \vdash \psi$ or $\vdash \psi$.*

Now we present some deduced rules in $\mathcal{OL}$.

(a) $\dfrac{\{\varphi\} \vdash \psi,\ \{\psi\} \vdash \sigma}{\{\varphi\} \vdash \sigma}$ (transitivity 1)

Consider the Cut $\dfrac{\Gamma \vdash \psi,\ \Delta \cup \{\psi\} \vdash \sigma}{\Gamma \cup \Delta \vdash \sigma}$ with $\Gamma = \{\varphi\}$ and $\Delta = \emptyset$.

(b) $\dfrac{\Gamma \vdash \psi,\ \{\psi\} \vdash \sigma}{\Gamma \vdash \sigma}$ (transitivity 2)

Consider the Cut $\dfrac{\Gamma \vdash \psi,\ \Delta \cup \{\psi\} \vdash \sigma}{\Gamma \cup \Delta \vdash \sigma}$ with $\Delta = \emptyset$.

(c) $\dfrac{\Gamma \vdash \psi,\ \Gamma \vdash \neg\psi}{\Gamma \vdash \varphi}$ (contradiction)

1. $\Gamma \vdash \psi$ premise
2. $\Gamma \vdash \neg\psi$ premise
3. $\Gamma \vdash \psi \wedge \neg\psi$ right conjunction in 1 and 2
4. $\{\psi \wedge \neg\psi\} \vdash \varphi$ explosion
5. $\Gamma \vdash \varphi$ (b) in 3 and 4.

(d) If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$

1. $\{\varphi\} \vdash \varphi$	auto-deductibility
2. $\Gamma \vdash \varphi$	monotonicity in 1.

(e) $\dfrac{\{\psi\} \vdash \varphi, \{\neg\psi\} \vdash \varphi}{\vdash \varphi}$ (proof by cases)

1. $\{\psi\} \vdash \varphi$	premise
2. $\{\neg\psi\} \vdash \varphi$	premise
3. $\{\neg\varphi\} \vdash \neg\psi$	contraposition in 1
4. $\{\neg\varphi\} \vdash \neg\neg\psi$	contraposition in 2
5. $\vdash \neg\neg\varphi$	absurdity in 3 and 4
6. $\vdash \varphi$	double negation in 5.

This system is particularly planned for derivations but not for proofs of theorems. However we can show some case of theorem.

(f) $\vdash \psi \vee \neg\psi$ (excluded middle)

1. $\{\varphi \wedge \neg\varphi\} \vdash \varphi$	simplification
2. $\{\varphi \wedge \neg\varphi\} \vdash \neg\varphi$	simplification
3. $\vdash \neg(\varphi \wedge \neg\varphi)$	absurdity in 1 and 2
4. $\vdash \neg\varphi \vee \neg\neg\varphi$	De Morgan in 3
5. $\vdash \psi \vee \neg\psi$	replacement in 4.

Goldblatt defined theorem in this logic as any formula φ such that $\psi \vee \neg\psi \vdash \varphi$ holds [9].

Proposition 2.5 $\{\varphi_1, \ldots, \varphi_n\} \vdash \psi \iff \varphi_1 \wedge \ldots \wedge \varphi_n \vdash \psi$.

Proof. ($\Rightarrow$) By $n-1$ applications of left conjunction.

($\Leftarrow$) By auto-deductibility we have $\{\varphi_i\} \vdash \varphi_i$, for $1 \leq i \leq n$. Then, by monotonicity $\{\varphi_1, \ldots, \varphi_n\} \vdash \varphi_i$, for $1 \leq i \leq n$. From that, applying right conjunction $n-1$ times we have $\{\varphi_1, \ldots, \varphi_n\} \vdash \varphi_1 \wedge \ldots \wedge \varphi_n$ and using the hypothesis and the transitivity 2 we have that $\{\varphi_1, \ldots, \varphi_n\} \vdash \psi$. ■

Proposition 2.6 *(Finite deductibility)* $\Sigma \vdash \psi \iff$ *there is* Σ_f *finite such that* $\Sigma_f \subseteq \Sigma$ *and* $\Sigma_f \vdash \psi$.

Proof. Each derivation is finite and uses only a finite number of formulas. ■

Corollary 2.7 $\Sigma \vdash \psi \iff$ *there are* $\varphi_1, \ldots, \varphi_n \in \Sigma$ *such that* $\varphi_1 \wedge \ldots \wedge \varphi_n \vdash \psi$.

Definition 2.8 *(Inconsistent and consistent sets) A set of formulas Σ is inconsistent if there is a formula ψ such that $\Sigma \vdash \psi \wedge \neg\psi$. The set Σ is consistent if it is not inconsistent.*

Definition 2.9 *(Deductive closure) The deductive closure of the set Σ is the set of all formulas derivable from Σ, that is, $\overline{\Sigma} = \{\varphi : \Sigma \vdash \varphi\}$.*

Of course $\Sigma \subseteq \overline{\Sigma}$.

Definition 2.10 *(Theory) Theory is a set of formulas deductively closed, that is, $\Sigma = \overline{\Sigma}$.*

3 Soundness

In this section we show that every derivation in $\mathcal{OL}$ is sound, that is, if we have a syntactical derivation $\Sigma \vdash \psi$, then we also have a consequence of ψ from Σ but in a semantic context.

As a first step we need to present this semantic consequence.

Definition 3.1 *(Restrict valuation) Let $\mathbb{L}$ be an ortholattice. A restrict valuation is a function $\breve{v} : Var(\boldsymbol{L}) \to \mathbb{L}$ that maps each variable of $\mathcal{OL}$ over an element of $\mathbb{L}$.*

Definition 3.2 *(Valuation) Valuation is a function $v : For(\boldsymbol{L}) \to \mathbb{L}$ that extends naturally and uniquely the function $\breve{v}$ as follows:*
(i) $v(p) = \breve{v}(p)$
(ii) $v(\neg\varphi) = v(\varphi)'$
(iii) $v(\varphi \wedge \psi) = v(\varphi) \curlywedge v(\psi)$.

Definition 3.3 *(Algebraic realization) Algebraic realization is a pair $(\mathbb{L}, v)$ such that $\mathbb{L}$ is an ortholattice and v is a valuation for $\mathcal{OL}$.*

Definition 3.4 *(Algebraic model) Let $\Gamma \subseteq For(\boldsymbol{L})$ and $(\mathbb{L}, v)$ an algebraic realization for $\mathcal{OL}$. Then $\mathcal{A} = (\mathbb{L}, v)$ is an algebraic model for Γ, or $\mathcal{A}$ satisfies Γ, if $v(\gamma) = 1$, for every $\gamma \in \Gamma$.*

We denote that $\mathcal{A} = (\mathbb{L}, v)$ is a model for Γ by $\mathcal{A} \vDash \Gamma$ and, in particular, if $\varphi \in For(\mathbf{L})$ and $v(\varphi) = 1$, then $\mathcal{A} \vDash \varphi$.

Definition 3.5 *(Validity in $\mathbb{L}$) A formula φ is valid in $\mathbb{L}$ if for every valuation v, the algebraic realization $\mathcal{A} = (\mathbb{L}, v)$ satisfies φ, that is, $\mathcal{A} \vDash \varphi$, for every valuation v.*

In this case we fix $\mathbb{L}$ but take any valuation v.

Definition 3.6 *(Valid formula) A formula φ is valid if it is valid in any algebraic realization $\mathcal{A}$.*

Now we do not fix any valuation v neither any ortholattice $\mathbb{L}$. We denote that φ is valid by $\vDash \varphi$.

We will denote any valid formula by $\top$, and any invalid formula by $\bot$. A formula is invalid if it is not valid in any algebraic realization.

Definition 3.7 *(Algebraic consequence relative to $\mathcal{A}$) Let $\Gamma \subseteq For(\boldsymbol{L})$ and $\mathcal{A} = (\mathbb{L}, v)$ an algebraic realization. A formula ψ is an algebraic consequence of Γ relative to $\mathcal{A}$, what is denoted by $\Gamma \vDash_{\mathcal{A}} \psi$, if:*

$$\textit{for any } b \in L, \textit{ if } b \leq v(\gamma) \textit{ for every } \gamma \in \Gamma, \textit{ then } b \leq v(\psi).$$

The idea is that $v(\psi)$ must be equal or bigger than the infimum of $\{v(\gamma) : \gamma \in \Gamma\}$. If $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$, then $\Gamma \vDash \psi \Leftrightarrow v(\gamma_1) \curlywedge \ldots \curlywedge v(\gamma_n) \leq v(\psi)$ and, in particular, $\{\gamma\} \vDash_{\mathcal{A}} \psi \iff v(\gamma) \leq v(\psi)$.

It is usual to define a similar consequence in the following way: [$*$] If $\Gamma \cup \{\psi\}$ is a set of formulas, then Γ implies ψ in the model $\mathcal{A}$, if $v_{\mathcal{A}}(\gamma) = 1$, for every $\gamma \in \Gamma$, then $v_{\mathcal{A}}(\psi) = 1$.

The above definition implies this condition [$*$], but they are not equivalent.

If we have some $\mathcal{A}$ in which $0 < v_{\mathcal{A}}(\psi) < v_{\mathcal{A}}(\gamma) < 1$, then, in accordance to [$*$] we have $\{\gamma\} \vDash \psi$, but it does not happen following the above definition of consequence.

The definition is perfect for the characterization of ortholattices.

Definition 3.8 *(Logical Consequence) A formula ψ is a logical consequence of Γ, or Γ implies ψ, what is denoted by $\Gamma \vDash \psi$, if for any algebraic realization $\mathcal{A}$, $\Gamma \vDash_{\mathcal{A}} \psi$.*

Now we can prove the Soundness Theorem.

Theorem 3.9 *If $\Gamma \subseteq For(\boldsymbol{L})$, then $\Gamma \vdash \gamma \Rightarrow \Gamma \vDash \gamma$.*

Proof. We need to show that each rule of $\mathcal{OL}$ preserves the validity.

Let $\mathcal{A} = (\mathbb{L}, v)$ be any algebraic realization. Then $\mathbb{L}$ is an ortholattice and each rule of $\mathcal{OL}$ is valid because:

(ROL_1): $v(\varphi) = v(\varphi)$.
(ROL_2) and ($0ROL_3$): $v(\varphi) = v(\neg\neg\varphi)$.
(ROL_4) and ($0ROL_5$): $v(\varphi \wedge \psi) = v(\varphi) \curlywedge v(\psi) \leq v(\varphi), v(\psi)$.

(ROL_6): $v(\varphi \wedge \neg\varphi) = v(\varphi) \curlywedge v(\varphi)' = 0 \leq v(\sigma)$, for any σ.

(ROL_7): $\Gamma \vDash \psi$, then for any $b \in L$, if $b \leq v(\gamma)$ for every $\gamma \in \Gamma$, then $b \leq v(\psi)$. Now, if we take $\Gamma \cup \Sigma$, the values of Σ do not invalidate the condition, because they may only vary with lower values. Hence $\Gamma \cup \Sigma \vDash \psi$.

(ROL_8): $\{\varphi\} \vDash \psi \Leftrightarrow v(\varphi) \leq v(\psi) \Leftrightarrow v(\psi)' \leq v(\varphi)' \Leftrightarrow \{\neg\psi\} \vDash \neg\varphi$.

(ROL_9): $\{\varphi, \psi\} \vDash \sigma \Leftrightarrow v(\varphi) \curlywedge v(\psi) \leq v(\sigma) \Leftrightarrow v(\varphi \wedge \psi) \leq v(\sigma) \Leftrightarrow \{\varphi \wedge \psi\} \vDash \sigma$.

(ROL_{10}): if $\Gamma \vDash \varphi$ and $\Sigma \cup \{\varphi\} \vDash \psi$, then then for any $b \in L$, if $b \leq v(\gamma)$ for every $\gamma \in \Gamma$ then $b \leq v(\varphi)$, and then for any $b \in L$, if $b \leq v(\sigma)$ for every $\sigma \in \Gamma \cup \{\varphi\}$ then $b \leq v(\psi)$. Thus, for any $b \in L$, if $b \leq v(\delta)$ for every $\delta \in \Gamma \cup \Sigma$ then $b \leq v(\psi)$, that is, $\Gamma \cup \Sigma \vDash \psi$.

(ROL_{11}): if $\{\psi\} \vDash \varphi$ and $\{\psi\} \vDash \neg\varphi$, then $v(\psi) \leq v(\varphi)$ and $v(\psi) \leq v(\neg\varphi)$, so $v(\psi) \leq v(\varphi) \curlywedge v(\varphi)' = 0$ and $v(\psi) = 0$. Thus $v(\neg\psi) = 1$ e hence $\vDash \neg\psi$.

(ROL_{12}): if $\Gamma \vDash \varphi$ and $\Gamma \vDash \psi$, then for any $b \in L$, if $b \leq v(\gamma)$ for every $\gamma \in \Gamma$ then $b \leq v(\varphi)$, and for any $b \in L$, if $b \leq v(\gamma)$ for every $\gamma \in \Gamma$, then $b \leq v(\psi)$. As $v(\varphi \wedge \psi) = v(\varphi) \curlywedge v(\psi)$, then for any $b \in L$, if $b \leq v(\gamma)$ for every $\gamma \in \Gamma$, then $b \leq v(\varphi \wedge \psi)$, that is, $\Gamma \vDash \varphi \wedge \psi$. ■

4 Completeness

Now we need to show that the set of logical consequences and derivable formulas are the same.

The proof of completeness for this logic was generally done using some Kripke model, as we can see for example in ([9], p. 26) and ([6], p. 181). Our proof below has a specifically algebraic character. Pavičić, [16] also presents an algebraic proof though less general than the following.

Definition 4.1 *(Full set) A set of formulas Δ is full if it is non-empty, consistent and holds:*

(i) *if $\varphi \in \Delta$ and $\{\varphi\} \vdash \psi$, then $\psi \in \Delta$;*

(ii) *if $\varphi, \psi \in \Delta$, then $\varphi \wedge \psi \in \Delta$.*

Proposition 4.2 *If Δ is full, then:*

(a) *$\varphi, \psi \in \Delta \Leftrightarrow \varphi \wedge \psi \in \Delta$;*

(b) *$\Delta \vdash \varphi \Leftrightarrow \varphi \in \Delta$;*

(c) *$\top \in \Delta$.*

Proof. *(a)* If $\varphi \wedge \psi \in \Delta$, as $\{\varphi \wedge \psi\} \vdash \varphi$ and $\{\varphi \wedge \psi\} \vdash \psi$, then by item (i) above $\varphi, \psi \in \Delta$.

(b) If $\Delta \vdash \varphi$, by Corollary 2.7, there are $\psi_1, \ldots, \psi_n \in \Delta$ such that $\{\psi_1 \wedge \ldots \wedge \psi_n\} \vdash \varphi$ and, by *(a)*, $\psi_1 \wedge \ldots \wedge \psi_n \in \Delta$. So $\varphi \in \Delta$.

If $\varphi \in \Delta$, by example (d) $\Delta \vdash \varphi$.

(c) As for any Δ, $\Delta \vdash \top$, then, by *(b)*, $\top \in \Delta$. ■

From item (b), we observe that any full set is a theory.

Proposition 4.3 *If Δ_1 and Δ_2 are full, then $\Delta_1 \cap \Delta_2$ is full.*
Proof. If Δ_1 and Δ_2 are consistent, then $\Delta_1 \cap \Delta_2$ is consistent.

If $\varphi \in \Delta_1 \cap \Delta_2$ and $\{\varphi\} \vdash \psi$, then $\varphi \in \Delta_1$ and $\{\varphi\} \vdash \psi$, e $\varphi \in \Delta_2$ and $\{\varphi\} \vdash \psi$. As Δ_1 and Δ_2 are full, then $\psi \in \Delta_1$ and $\psi \in \Delta_2$. Hence $\psi \in \Delta_1 \cap \Delta_2$.

If $\varphi, \psi \in \Delta_1 \cap \Delta_2$, then $\varphi, \psi \in \Delta_1$ and $\varphi, \psi \in \Delta_2$. As Δ_1 and Δ_2 are full, then $\varphi \wedge \psi \in \Delta_1$ and $\varphi \wedge \psi \in \Delta_2$. Finally, $\varphi \wedge \psi \in \Delta_1 \cap \Delta_2$. ■

Proposition 4.4 $\Gamma \vdash \varphi \iff \varphi$ *belongs to every full extension of* Γ.
Proof. ($\Rightarrow$) Suppose that $\Gamma \vdash \varphi$ and Δ is a full extension of Γ. Then, by Corollary 2.7, there are $\varphi_1, \ldots, \varphi_n \in \Gamma \subseteq \Delta$ such that $\{\varphi_1 \wedge \ldots \wedge \varphi_n\} \vdash \varphi$. Moreover, by Definition 4.1 (ii) and (i), $\varphi_1 \wedge \ldots \wedge \varphi_n \in \Delta$ and hence $\varphi \in \Delta$.

($\Leftarrow$) By contrapositive, suppose that Γ is consistent and $\Gamma \nvdash \varphi$.

Thus $\varphi \notin \overline{\Gamma}$ and, of course, $\Gamma \subseteq \overline{\Gamma}$. So, we show that $\overline{\Gamma}$ is full.

Since Γ is consistent, then $\overline{\Gamma}$ is consistent and $\overline{\Gamma} \neq \emptyset$.

Now:

(i) suppose $\psi \in \overline{\Gamma}$ and $\{\psi\} \vdash \delta$. Then $\Gamma \vdash \psi$ and $\{\psi\} \vdash \delta$ and so, by example (b), $\Gamma \vdash \delta$ and hence $\delta \in \overline{\Gamma}$.

(ii) if $\psi, \delta \in \overline{\Gamma}$, then there are $\psi_1 \wedge \ldots \wedge \psi_n$, $\delta_1 \wedge \ldots \wedge \delta_m \in \Gamma$ such that $\{\psi_1 \wedge \ldots \wedge \psi_n\} \vdash \psi$ and $\{\delta_1 \wedge \ldots \wedge \delta_m\} \vdash \delta$. Thus, by monotonicity and right conjunction, $\{\psi_1 \wedge \ldots \wedge \psi_n \wedge \delta_1 \wedge \ldots \wedge \delta_m\} \vdash \psi \wedge \delta$. Therefore, $\Gamma \vdash \psi \wedge \delta$ and $\psi \wedge \delta \in \overline{\Gamma}$.

Hence $\overline{\Gamma}$ is full. ■

Definition 4.5 *(Compatible sets) The sets Δ and Λ are compatible if there is no formula ψ such that $\Delta \vdash \psi$ and $\Lambda \vdash \neg\psi$.*

The next result is more properly an observation.

Proposition 4.6 *If Δ and Λ are compatible, then for every formula ψ, if $\Delta \vdash \psi$, then $\Lambda \nvdash \neg\psi$.*

Theorem 4.7 *If $\Delta \nvdash \neg\varphi$, then there exists Λ compatible with Δ such that $\Lambda \vdash \varphi$.*

Proof. Suppose that $\Delta \nvdash \neg\varphi$. If $\Lambda = \{\varphi\}$, by autodeductibility, $\Lambda \vdash \varphi$ and Δ and Λ are compatible, for on the contrary there is some formula σ such that $\Delta \vdash \neg\sigma$ and $\Lambda \vdash \sigma$. Then $\{\varphi\} \vdash \sigma$ and by contraposition $\{\neg\sigma\} \vdash \neg\varphi$. As $\Delta \vdash \neg\sigma$, by cut, $\Delta \vdash \neg\varphi$. ■

Theorem 4.8 *Every consistent set Γ is included in a full set Λ.*

Proof. We take an enumeration $\psi_0, \psi_1, \psi_2, \ldots$ of $For(\mathbf{L})$ and construct a sequence of sets $\Lambda_i, i \in \mathbb{N}$, of compatible sets as in the previous theorem in the following way.

In the first step $\Lambda_0 = \Gamma$. So, if $\Lambda_n \vdash \neg\psi_n$, then $\Lambda_{n+1} = \Lambda_n \cup \{\neg\psi_n\}$ and if $\Lambda_n \nvdash \neg\psi_n$, then $\Lambda_{n+1} = \Lambda_n \cup \{\psi_n\}$. Thus, each set Λ_n is compatible with every previous set in the sequence and, by definition of compatible sets, they are consistent.

Finally, we take $\Lambda = \cup\Lambda_i, i \in \mathbb{N}$. This set is full, compatible with Γ and $\Gamma \subseteq \Lambda$. ■

Now we must construct a canonical algebraic realization for $\mathcal{OL}$. Its domain is the set $\mathbf{T}$ of all full theories of $\mathcal{OL}$.

On $\mathbf{T}$ we need to determine a structure of an ortholattice.

Definition 4.9 *(Structure of full sets) For $\varphi, \psi \in For(\boldsymbol{L})$, we define $\widehat{\ }: For(\boldsymbol{L}) \to \mathcal{P}(\boldsymbol{T})$:*

(i) $\widehat{\varphi} = \{\Delta \in \boldsymbol{T} : \Delta \vdash \varphi\}$
(ii) $\widehat{\bot} = \emptyset$
(iii) $\widehat{\top} = \boldsymbol{T}$
(iv) $\widehat{\varphi} \curlywedge \widehat{\psi} = \widehat{\varphi} \cap \widehat{\psi}$
(v) $\widehat{\varphi}\,' = \{\Delta \in \boldsymbol{T} : \Delta$ *is incompatible with* $\widehat{\varphi}\}$.

Of course, for every $\Delta \in \mathbf{T}$, $\bot \notin \Delta$. On the other side, $\top$ belongs to all full sets. The conjunction coincides with set intersection, but the negation is not the set complement, because we would have a Boolean algebra with the classical negation. The classical complementation is a particular case of ortholattice complementation, however the quantum negation is weaker than the classical one.

Lemma 4.10 $\{\varphi\} \vdash \psi \Leftrightarrow \widehat{\varphi} \subseteq \widehat{\psi}$.

Proof. $(\Rightarrow)$ If $\Delta \in \widehat{\varphi}$, then $\Delta \vdash \varphi$. Since $\{\varphi\} \vdash \psi$, then $\Delta \vdash \psi$ and hence $\Delta \in \widehat{\psi}$.

$(\Leftarrow)$ If $\{\varphi\} \nvdash \psi$, then there exists $\Delta \in \mathbf{T}$ such that $\Delta \vdash \varphi$ and $\Delta \nvdash \psi$. Thus, $\Delta \in \widehat{\varphi}$, but $\Delta \notin \widehat{\psi}$. So $\widehat{\varphi} \nsubseteq \widehat{\psi}$. ■

Now we need to prove the following important result.

Proposition 4.11 *The structure $\langle \boldsymbol{T}, \subseteq, {}', \curlywedge, \widehat{\bot}\rangle$ is an ortholattice.*
Proof. As the relation $\vdash$ is reflexive, transitive and antisymmetric, from the previous lemma, it follows that the relation $\subseteq$ is a partial order on $\mathbf{T}$. Besides, $\widehat{\bot}$ and $\widehat{\top}$ are the 0 and 1 on $\mathbf{T}$.

Then $\langle \mathbf{T}, \subseteq, {}', \curlywedge, \widehat{\bot}\rangle$ is a complemented partial order with 0 and 1, because: (i) $\widehat{\psi} \curlywedge \widehat{\neg\psi} = \widehat{\bot}$ and (ii) $\widehat{\psi} \curlyvee \widehat{\neg\psi} = \widehat{\top}$.

(i) $\widehat{\bot} = \emptyset \subseteq \widehat{\psi} \cap \widehat{\neg\psi} = \widehat{\psi} \curlywedge \widehat{\neg\psi}$. And $\Delta \in \widehat{\psi} \curlywedge \widehat{\neg\psi} \Rightarrow \Delta \in \widehat{\psi}$ and $\Delta \in \widehat{\neg\psi} \Rightarrow \Delta \vdash \psi$ and $\Delta \vdash \neg\psi \Rightarrow \Delta \vdash \bot$. So $\widehat{\psi} \curlywedge \widehat{\neg\psi} \subseteq \widehat{\bot}$.

(ii) $\widehat{\top} \subseteq \widehat{\psi}$ and $\widehat{\top} \subseteq \widehat{\neg\psi} \Rightarrow \widehat{\top} \subseteq \widehat{\psi} \curlyvee \widehat{\neg\psi} = \widehat{\psi} \curlyvee \widehat{\psi}\,'$. And $\Delta \in \widehat{\psi} \curlyvee \widehat{\neg\psi} \Rightarrow \Delta \in \widehat{\psi}$ or $\Delta \in \widehat{\neg\psi} \Rightarrow \Delta \vdash \psi$ or $\Delta \vdash \neg\psi \Rightarrow \Delta \vdash \psi \vee \neg\psi \Rightarrow \Delta \vdash \top \Leftrightarrow \Delta \in \widehat{\top}$. So $\widehat{\psi} \curlyvee \widehat{\neg\psi} \subseteq \widehat{\top}$.

Now we need to show that $'$ is an involution.

(iii) Suppose that $\widehat{\varphi} \neq \widehat{\varphi}\,''$. Thus either there is $\Delta_1 \in \mathbf{T}$ such that $\Delta_1 \vdash \neg\neg\varphi$ but $\Delta_1 \nvdash \varphi$, or there is $\Delta_2 \in \mathbf{T}$ such that $\Delta_2 \vdash \varphi$ but $\Delta_2 \nvdash \neg\neg\varphi$. We shall analyse only one case. As Δ_2 is full and $\{\varphi\} \vdash \neg\neg\varphi$, then $\Delta_2 \vdash \neg\neg\varphi$. In any case we have a contradiction.

(iv) By Lemma 5.10, $\widehat{\varphi} \subseteq \widehat{\psi} \Leftrightarrow \{\varphi\} \vdash \psi$ and, by Contraposition, $\{\varphi\} \vdash \psi \Leftrightarrow \{\neg\psi\} \vdash \neg\varphi$. Again by lemma $\widehat{\varphi} \subseteq \widehat{\psi} \Leftrightarrow \widehat{\neg\psi} \subseteq \widehat{\neg\varphi} \Leftrightarrow \widehat{\psi}\,' \subseteq \widehat{\varphi}\,'$. ■

Definition 4.12 *(Canonical valuation) A canonical valuation is any valuation $[.] : For(\boldsymbol{L}) \to \mathcal{P}(T)$ such that:*

(i) $[p] := \{\Delta \in \boldsymbol{T} : p \in \Delta\} = \widehat{p}$.

Proposition 4.13 *For every $\varphi \in For(\boldsymbol{L})$, it follows that $[\varphi] = \widehat{\varphi}$.*
Proof. By induction on the complexity of φ.

If φ is a propositional variable, then $[p] = \widehat{p}$, by the above definition.

If φ is of the type $\neg\psi$, then by induction hypotheses, $[\psi] = \widehat{\psi}$. So $[\varphi] = [\neg\psi] = [\psi]' = \widehat{\psi}\,' = \{\Delta \in \mathbf{T} : \Delta \textit{ is incompatible with } \widehat{\psi}\} = \{\Delta \in \mathbf{T} : \Delta \vdash \neg\psi\} = \widehat{\neg\psi} = \widehat{\varphi}$.

If φ is of the type $\psi \wedge \sigma$, then by induction hypotheses, $[\psi] = \widehat{\psi}$ and $[\sigma] = \widehat{\sigma}$. So $[\varphi] = [\psi \wedge \sigma] = [\psi] \curlywedge [\sigma] = \widehat{\psi} \cap \widehat{\sigma} = \widehat{\varphi}$. ■

Theorem 4.14 *(Strong completeness) If $\Gamma \vDash \psi$, then $\Gamma \vdash \psi$.*
Proof. If $\Gamma \nvdash \psi$, then $\Gamma \cup \{\neg\psi\}$ is consistent. By Theorem 4.8, there exists a full set Λ such that $\Gamma \cup \{\neg\psi\} \subseteq \Lambda$. As Λ is full and $\neg\psi \in \Lambda$, then $\Lambda \vdash \neg\psi$ and so $\Lambda \in \widehat{\neg\psi}$. Thus $\Lambda \vDash \neg\psi$. As Λ is full, then $\Lambda \nvDash \psi$ and considering that $\Gamma \subseteq \Lambda$, then $\Gamma \nvDash \psi$. ■

In this view the compactness is very simple.

Corollary 4.15 *(Compactness) If every finite $\Gamma_f \subseteq \Gamma$ has a model, then Γ has a model.*
Proof. If Γ does not have a model, then for every $\Delta \in \mathbf{T}$, it follows that $\Gamma \nsubseteq \Delta$. By Theorem 4.8, Γ is inconsistent. Hence, there is a formula ψ such that $\Gamma \vdash \psi$ and $\Gamma \vdash \neg\psi$, that is, there is a finite set $\Gamma_f \subseteq \Gamma$ such that $\Gamma_f \vdash \psi$ and $\Gamma_f \vdash \neg\psi$. Thus, the set Γ_f does not have a model. ■

5 Final remarks

We presented the ortholattices and a proof of adequacy between the algebraic ortholattices and the logic of ortholattices $\mathcal{OL}$ using only algebraic tools.

In the next steps we will try to include a conditional in $\mathcal{OL}$ and consider some specifications of ortholattices given by the introduction of new algebraic axioms. Of course, we must observe how the logical systems follow the algebraic inclusions.

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