

Classifying Toposes for Some Theories of \mathcal{C}^∞ –Rings

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Abstract

In this paper we present classifying toposes for the following theories: the theory of \mathcal{C}^∞ –rings, the theory of local \mathcal{C}^∞ –rings and the theory of von Neumann regular \mathcal{C}^∞ –rings. The classifying toposes for the first two theories were stated without proof by Ieke Moerdijk and Gonzalo Reyes on the page 366 of [16], where they assert that the topos $\mathbf{Set}^{\mathcal{C}^\infty \mathbf{Rng}_{\text{fp}}}$ classifies the theory of \mathcal{C}^∞ –rings and that the smooth Zariski topos classifies the theory of local \mathcal{C}^∞ –rings. We begin by constructing a classifying topos for the theory of \mathcal{C}^∞ –rings, by mimicking the construction of a classifying topos for the theory of commutative unital rings given in [12], and then we prove that the smooth Zariski topos classifies the theory of local \mathcal{C}^∞ –rings. We also give a description of the classifying topos for the theory of von Neumann regular \mathcal{C}^∞ –rings.

Keywords: Classifying Toposes, \mathcal{C}^∞ –rings, local \mathcal{C}^∞ –rings, von Neumann regular \mathcal{C}^∞ –rings.

Introduction

An \mathbb{R} –algebra A in a category with finite limits, \mathcal{C} , may be regarded as a finite product preserving functor from the category \mathbf{Pol} , whose objects are given by $\text{Obj}(\mathbf{Pol}) = \{\mathbb{R}^n | n \in \mathbb{N}\}$, and whose morphisms are given by polynomial functions between them, $\text{Mor}(\mathbf{Pol}) = \{\mathbb{R}^m \xrightarrow{p} \mathbb{R}^n | m, n \in \mathbb{N}, p \text{ polynomial}\}$, to \mathcal{C} , that is:

$$A : \mathbf{Pol} \rightarrow \mathcal{C}.$$

In this sense, an \mathbb{R} –algebra A is a functor which interprets all polynomial maps $p : \mathbb{R}^m \rightarrow \mathbb{R}^n$, for $m, n \in \mathbb{N}$. In this vein, one may define a \mathcal{C}^∞ –ring as a finite product preserving functor from the category \mathcal{C}^∞ , whose objects are given by $\text{Obj}(\mathcal{C}^\infty) = \{\mathbb{R}^n | n \in \mathbb{N}\}$ and whose morphisms are given by \mathcal{C}^∞ –functions between them, $\text{Mor}(\mathcal{C}^\infty) = \{\mathbb{R}^m \xrightarrow{f} \mathbb{R}^n | m, n \in \mathbb{N}, f \text{ smooth function}\}$, *i.e.*,

$$A : \mathcal{C}^\infty \rightarrow \mathcal{C}.$$

According to I. Moerdijk and G. Reyes in [14], the original motivation to introduce and study \mathcal{C}^∞ -rings was to construct topos-models for Synthetic Differential Geometry. Their introduction circumvent some obstacles for a synthetic framing for Differential Geometry in **Set**, like, for instance, the lack, in the category of smooth manifolds, of finite inverse limits (in particular, even binary pullbacks of \mathcal{C}^∞ -manifolds are not manifolds, unless a condition of transversality is fulfilled) and the absence of a convenient language to deal explicitly and directly with structures in the “infinitely small” level (cf. [16]). The existence of nilpotent elements, which provides us with a language that legitimates the use of geometric intuition does not come for free: the essential Kock-Lawvere axiom and its consequences, for example, are not compatible with the principle of the excluded middle (see [11]). Thus, in order to deal with \mathcal{C}^∞ -rings one must give up on Classical Logic, and this necessarily leads us to the need for “toposes” - which can be seen as “mathematical worlds” in which one has an intuitionistic logic.

The theory of \mathcal{C}^∞ -rings can be interpreted in any category \mathcal{C} with finite products. However, as we consider theories of \mathcal{C}^∞ -rings that require its models to satisfy axioms with connectives such as “disjunctions” (which is the case for the theory of local \mathcal{C}^∞ -rings), we need “richer categorical constructions” (such as the possibility of forming unions of subobjects) in order to interpret them meaningfully in any topos.

It is a well-known result that some types of first order theories - depending on the language and on the structure of their axioms always have a classifying topos (cf. [13]). Among the first order theories which have a classifying topos we find the so-called “geometric theories”, *i.e.*, theories (possibly infinitary and poli-sorted) whose axioms consist of implications between geometric formulas.

In this paper we are concerned with a concrete description of the classifying topoi of the (equational) theory of \mathcal{C}^∞ -rings, the (geometric) theory of the local \mathcal{C}^∞ -rings and the (equational) theory of von Neumann regular \mathcal{C}^∞ -rings. We present a step-by-step construction of such topoi, mimicking the construction of the classifying topoi for the theory of rings and for the theory of local rings given in [12] with some adaptations.

Overview of the Paper

The organization of this paper is as follows.

In the first section we present some concepts and preliminary results on categorial logic, classifying toposes and \mathcal{C}^∞ -rings.

In section 2 we give a comprehensive description of the classifying topos for the theory of \mathcal{C}^∞ -rings as a presheaf category. In the third section we give a detailed description of the smooth Zariski (Grothendieck) topology and its corresponding sheaf topos as the classifying topos for the theory of local \mathcal{C}^∞ -rings.

In the final section we introduce the notion of a von Neumann regular \mathcal{C}^∞ -ring along with some of its characterizations and we describe the classifying topos for the (first-order) theory of von Neumann regular \mathcal{C}^∞ -rings. We also present some related topics which can be developed in future works.

1 Preliminaries

1.1 On \mathcal{C}^∞ -rings

In order to formulate and study the concept of \mathcal{C}^∞ -ring, we are going to use a first order language \mathcal{L} with a denumerable set of variables ($\text{Var}(\mathcal{L}) = \{x_1, x_2, \dots, x_n, \dots\}$) whose nonlogical symbols are the symbols of \mathcal{C}^∞ -functions from \mathbb{R}^m to \mathbb{R}^n , with $m, n \in \mathbb{N}$, *i.e.*, the non-logical symbols consist only of function symbols, described as follows:

For each $n \in \mathbb{N}$, the n -ary **function symbols** of the set $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$, *i.e.*, $\mathcal{F}_{(n)} = \{f^{(n)} \mid f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})\}$. So the set of function symbols of our language is given by:

$$\mathcal{F} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_{(n)} = \bigcup_{n \in \mathbb{N}} \mathcal{C}^\infty(\mathbb{R}^n)$$

Note that our set of constants is identified with the set of all 0-ary function symbols, *i.e.*, $\mathcal{C} = \mathcal{F}_{(0)} = \mathcal{C}^\infty(\mathbb{R}^0) \cong \mathcal{C}^\infty(\{*\})$.

The terms of this language are defined, in the usual way, as the smallest set which comprises the individual variables, constant symbols and n -ary function symbols followed by n terms ($n \in \mathbb{N}$).

Apart from the functorial definition we gave in the introduction, we have many equivalent descriptions. We focus, first, in the following description of a \mathcal{C}^∞ -ring in **Set**.

Definition 1.1 A \mathcal{C}^∞ -**structure** on a set A is a pair $\mathfrak{A} = (A, \Phi)$, where:

$$\begin{aligned} \Phi : \bigcup_{n \in \mathbb{N}} \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) &\rightarrow \bigcup_{n \in \mathbb{N}} \text{Func}(A^n; A) \\ (f : \mathbb{R}^n \xrightarrow{\mathcal{C}^\infty} \mathbb{R}) &\mapsto \Phi(f) := (f^A : A^n \rightarrow A) \end{aligned}$$

that is, Φ interprets the **symbols**¹ of all smooth real functions of n variables as n -ary function symbols on A .

¹here considered simply as syntactic symbols rather than functions.

We call a C^∞ -structure $\mathfrak{A} = (A, \Phi)$ a C^∞ -**ring** if it preserves projections and all equations between smooth functions. We have the following:

Definition 1.2 Let $\mathfrak{A} = (A, \Phi)$ be a C^∞ -structure. We say that \mathfrak{A} (or, when there is no danger of confusion, A) is a C^∞ -**ring** if the following is true:

- Given any $n, k \in \mathbb{N}$ and any projection $p_k : \mathbb{R}^n \rightarrow \mathbb{R}$, we have:

$$\mathfrak{A} \models (\forall x_1) \cdots (\forall x_n) (p_k(x_1, \dots, x_n) = x_k)$$

- For every $f, g_1, \dots, g_n \in C^\infty(\mathbb{R}^m, \mathbb{R})$ with $m, n \in \mathbb{N}$, and every $h \in C^\infty(\mathbb{R}^n, \mathbb{R})$ such that $f = h \circ (g_1, \dots, g_n)$, one has:

$$\mathfrak{A} \models (\forall x_1) \cdots (\forall x_m) (f(x_1, \dots, x_m) = h(g(x_1, \dots, x_m), \dots, g_n(x_1, \dots, x_m)))$$

Definition 1.3 Let (A, Φ) and (B, Ψ) be two C^∞ -rings. A function $\varphi : A \rightarrow B$ is called a **morphism of C^∞ -rings** or **C^∞ -homomorphism** if for any $n \in \mathbb{N}$ and any $f : \mathbb{R}^n \xrightarrow{C^\infty} \mathbb{R}$ the following diagram commutes:

$$\begin{array}{ccc} A^n & \xrightarrow{\varphi^{(n)}} & B^n \\ \Phi(f) \downarrow & & \downarrow \Psi(f) \\ A & \xrightarrow{\varphi} & B \end{array}$$

i.e., $\Psi(f) \circ \varphi^{(n)} = \varphi \circ \Phi(f)$.

Remark 1.4 Observe that C^∞ -structures, together with their morphisms compose a category, that we denote by $C^\infty\mathbf{Str}$, and that C^∞ -rings, together with all the C^∞ -homomorphisms between C^∞ -rings compose a full subcategory of $C^\infty\mathbf{Rng}$. In particular, since $C^\infty\mathbf{Rng}$ is a “variety of algebras” (it is a class of C^∞ -structures which satisfy a given set of equations), it is closed under substructures, homomorphic images and products, by **Birkhoff’s HSP Theorem**. Moreover:

- $C^\infty\mathbf{Rng}$ is a concrete category and the forgetful functor, $U : C^\infty\mathbf{Rng} \rightarrow \mathbf{Set}$ creates directed inductive colimits;
- Each set X freely generates a C^∞ -ring, in particular $C^\infty(\mathbb{R}^n, \mathbb{R})$ is the free C^∞ -ring on n generators, $n \in \mathbb{N}$;
- The congruences of C^∞ -rings are classified by their “ring-theoretical” ideals;
- Every C^∞ -ring is the homomorphic image of some free C^∞ -ring determined by some set, being isomorphic to the quotient of a free C^∞ -ring by some ideal.

Within the category of C^∞ -rings, we have two special subcategories, that we define in the sequel.

Definition 1.5 A C^∞ -ring A is **finitely generated** whenever there is some $n \in \mathbb{N}$ and some ideal $I \subseteq C^\infty(\mathbb{R}^n)$ such that $A \cong \frac{C^\infty(\mathbb{R}^n)}{I}$. The category of all finitely generated C^∞ -rings is denoted by $C^\infty\mathbf{Rng}_{\text{fg}}$.

Definition 1.6 A C^∞ -ring is **finitely presented** whenever there is some $n \in \mathbb{N}$ and some finitely generated ideal $I \subseteq C^\infty(\mathbb{R}^n)$ such that $A \cong \frac{C^\infty(\mathbb{R}^n)}{I}$.

Whenever A is a finitely presented C^∞ -ring, there is some $n \in \mathbb{N}$ and some $f_1, \dots, f_k \in C^\infty(\mathbb{R}^n)$ such that:

$$A = \frac{C^\infty(\mathbb{R}^n)}{\langle f_1, \dots, f_k \rangle}$$

The category of all finitely presented C^∞ -rings is denoted by $C^\infty\mathbf{Rng}_{\text{fp}}$

These categories are closed under initial objects, binary coproducts and binary coequalizers. Thus, they are finitely co-complete categories, that is, they have all finite colimits (for a proof of this fact we refer to the chapter 1 of [2]).

Since $C^\infty\mathbf{Rng}_{\text{fp}}$ has all finite colimits, it follows that $C^\infty\mathbf{Rng}_{\text{fp}}^{\text{op}}$ has all finite limits.

1.2 Categorical Logic and classifying topoi

In this subsection we list the main logical-categorical notions and results that we will need in the sequel of this work. The main references here are [12], [7], [6] and [13].

(I) Sketches and their models:

- A (small) sketch is a 4-tuple $\mathcal{S} = (G, D, P, I)$ ([6]), where G is a (small) oriented graph; D is a (set)class of small (non-commutative) diagrams over G ; P is a (set)class of (non-commutative) cones over G ; I is a (set)class of (non-commutative) co-cones over G . \mathcal{S} is a geometric sketch if P is a set of cones over G with finite basis. Each (small) category \mathcal{C} determines a (small) sketch: $\text{sk}(\mathcal{C}) = (|\mathcal{C}|, D_{\mathcal{C}}, P_{\mathcal{C}}, I_{\mathcal{C}})$, where $|\mathcal{C}|$ is the underlying graph of the category, $D_{\mathcal{C}}$ is the class of all small commutative over \mathcal{C} , $P_{\mathcal{C}}$ is the class of all small limit cones over \mathcal{C} , $I_{\mathcal{C}}$ is the class of all small colimit co-cones over \mathcal{C} . A sketch $\mathcal{S} = (G, D, P, I)$ is called a $(\mathcal{P}, \mathcal{I})$ -type if the base of all cones in P are in the class \mathcal{P} and if the base of all co-cones in I are in the class \mathcal{I} .

- A morphism of sketches $\mathcal{S} \rightarrow \mathcal{S}'$ is a homomorphism of the underlying graphs that preserves all the given structures. This determines a (very large) category SK.

- A model of a sketch \mathcal{S} in a category \mathcal{C} is a morphism of sketches $\mathcal{S} \rightarrow \text{sk}(\mathcal{C})$. We will denote $\text{Mod}(\mathcal{S}, \mathcal{C})$ the category whose objects are the models of \mathcal{S} into the category \mathcal{C} and the arrows are the natural transformations between the models (this makes sense since \mathcal{C} is a category). Many usual categories of (first-order, but not necessarily finitary) mathematical structures \mathcal{K} can be described as $\mathcal{K} \simeq \text{Mod}(\mathcal{S}, \mathbf{Set}) = \text{SK}(\mathcal{S}, \text{sk}(\mathbf{Set}))$ for some small sketch \mathcal{S} ; for instance: groups and their homomorphisms, rings and their homomorphisms, fields and their homomorphisms, local rings and local homomorphisms, σ -boolean algebras and their homomorphisms, Banach spaces and linear contractions.

- Every small sketch \mathcal{S} of $(\mathcal{P}, \mathcal{I})$ -type has a “canonical” $(\mathcal{P}, \mathcal{I})$ -model $M : \mathcal{S} \rightarrow \text{sk}(\hat{\mathcal{S}})$, where $\hat{\mathcal{S}}$ is a \mathcal{P} -complete and \mathcal{I} -cocomplete category called “the $(\mathcal{P}, \mathcal{I})$ -theory of \mathcal{S} ”. That is, it has all limits of the type occurring. This means that for each category \mathcal{C} that is \mathcal{P} -complete and \mathcal{I} -cocomplete composing with M yields an equivalence of categories $\text{Func}_{(\mathcal{P}, \mathcal{I})}(\hat{\mathcal{S}}, \mathcal{C}) \xrightarrow{\sim} \text{Mod}(\mathcal{S}, \mathcal{C}) = \text{SK}(\mathcal{S}, \text{sk}(\mathcal{C}))$, where $\text{Func}_{(\mathcal{P}, \mathcal{I})}(\hat{\mathcal{S}}, \mathcal{C})$ is the full subcategory of $\text{Func}(\hat{\mathcal{S}}, \mathcal{C})$, of all functors that preserves \mathcal{P} -limits and \mathcal{I} -colimits. The $(\mathcal{P}, \mathcal{I})$ -theory $\hat{\mathcal{S}}$ is unique up to “equivalence of categories”.

(II) Grothendieck Topoi and geometric morphisms:

- A (small) site is a pair (\mathcal{C}, J) formed by a (small) category \mathcal{C} and a Grothendieck (pre)topology J on \mathcal{C} , i.e. a map $C \in \text{Obj}(\mathcal{C}) \mapsto J(C)$ where $f \in J(C)$ is a small family of \mathcal{C} -arrows $\mathcal{F} = \{f_i : A_i \rightarrow C\}_{i \in I}$ that satisfies: the isomorphism axiom; stability axiom and transitivity axiom ([12]). The usual notion of covering by opens in a topological space X provides a site $(\text{Open}(X), J)$.

- Similar to the case of (pre)sheaves over a topological space it can be defined in general the (pre)sheaves category: $\text{Sh}(\mathcal{C}, J) \hookrightarrow \mathbf{Set}^{\mathcal{C}^{\text{op}}}$ and the sheafification (left adjoint) functor $a : \mathbf{Set}^{\mathcal{C}^{\text{op}}} \rightarrow \text{Sh}(\mathcal{C}, J) : \text{determines a geometric morphism.}$

- A Grothendieck topos \mathcal{E} is a category that is equivalent to the category of sheaves over a small site (\mathcal{C}, J) , $\mathcal{E} \simeq \text{Sh}(\mathcal{C}, J) \hookrightarrow \mathbf{Set}^{(\mathcal{C}^{\text{op}})}$.

- A geometric morphism between the Grothendieck topoi $\mathcal{E}, \mathcal{E}'$, $f : \mathcal{E} \rightarrow \mathcal{E}'$, is a functor $f^* : \mathcal{E}' \rightarrow \mathcal{E}$ that preserves small colimits and is left exact (i.e. it preserves finite limits). Equivalently a geometric morphism $\mathcal{E} \rightarrow \mathcal{E}'$ a is an equivalent class of adjoint functors

$$\mathcal{E} \begin{matrix} \xrightarrow{f_*} \\ \xleftarrow{f^*} \end{matrix} \mathcal{E}'$$

where f^* is left exact and left adjoint to f_* , and $(f^*, f_*) \equiv (g^*, g_*)$ iff $f^* = g^*$ (and thus $f_* \cong g_*$). If (\mathcal{C}, J) is a small site, the “sheafification (left adjoint)

functor” $a : \mathbf{Set}^{\mathcal{C}^{\text{op}}} \rightarrow \text{Sh}(\mathcal{C}, J)$ determines a geometric morphism $\text{Sh}(\mathcal{C}, J) \rightarrow \mathbf{Set}^{\mathcal{C}^{\text{op}}}$.

- If \mathcal{E}, \mathcal{F} are Grothendieck topoi, we denote $\text{Geom}(\mathcal{F}, \mathcal{E}) \hookrightarrow \text{Func}(\mathcal{E}, \mathcal{F})$ the full subcategory of the category of functors and natural transformations formed by the (left adjoint part) of geometric morphisms $\mathcal{F} \rightarrow \mathcal{E}$.

(III) (Functorial) Theories:

- A mathematical theory T will be called a *functorial* mathematical theory, when there is a small category \mathcal{C}_T such that the category of models of this theory in a Grothendieck topos \mathcal{F} , $\text{Mod}_{\mathcal{F}}(T)$ is (naturally) equivalent to a full subcategory of $\text{Hom}_T(\mathcal{C}_T, \mathcal{F}) \hookrightarrow \text{Func}(\mathcal{C}_T, \mathcal{E})$. This category \mathcal{C}_T is unique up to equivalence.

- Let \mathcal{C} be a small category with finite products and consider the (functorial) theory of finite product preserving functors on \mathcal{C} , i.e. $\mathcal{C}_T = \mathcal{C}$ and $\text{Mod}_T(\mathcal{E}) = \text{Hom}_T(\mathcal{C}_T, \mathcal{E}) = \text{Prod}_{\text{fin}}(\mathcal{C}_T, \mathcal{E}) \hookrightarrow \text{Func}(\mathcal{C}, \mathcal{E})$.

- Let \mathcal{C} be a small left exact category (i.e. \mathcal{C} has all finite limits) and consider the (functorial) theory of left exact functors (= finite limits preserving functors) on \mathcal{C} , i.e. $\mathcal{C}_T = \mathcal{C}$ and $\text{Mod}_T(\mathcal{E}) = \text{Hom}_T(\mathcal{C}_T, \mathcal{E}) = \mathbf{Lex}(\mathcal{C}_T, \mathcal{E}) \hookrightarrow \text{Func}(\mathcal{C}, \mathcal{E})$.

- Examples of functorial mathematical theories are given by the theories $\hat{\mathcal{S}}$ associated to small sketches $\mathcal{S} = (G, D, P, I)$ (see (I) above).

- To each geometric/coherent first-order theory in the infinitary language $L_{\infty\omega}$ can be associate a small "syntactical" category \mathcal{C}_T in such a way to determine a functorial theory ([13]).

(IV) Classifying topoi:

- Let T be a functorial mathematical theory. T admits a classifying topos when there are (i) a Grothendieck topos $\mathcal{E}(T)$; (ii) a model $M : \mathcal{C}_T \rightarrow \mathcal{E}(T)$; that are (2-)universal in the following sense: given a Grothendieck topos \mathcal{F} , composing M with the left adjoint part of the geometric morphism yields an equivalence of categories $\text{Geom}(\mathcal{F}, \mathcal{E}[T]) \xrightarrow{\cong} \text{Hom}_T(\mathcal{C}_T, \mathcal{F})$. The topos $\mathcal{E}[T]$ is called *the* classifying topos of the theory T and the model M is called *the* generic model of the theory T .

- Each classifying topos of a functorial mathematical theory determines an equivalence of categories $\text{Geom}(\mathcal{F}, \mathcal{E}[T]) \simeq \text{Mod}_{\mathcal{F}}(T)$, for each Grothendieck topos \mathcal{F} . When a functorial mathematical theory admits is a classifying topos, it is unique up to equivalence of categories.

- Let \mathcal{C} be a small left exact category, then the theory of left exact functors on \mathcal{C} admits the presheaves category $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$ as a classifying topos and the Yoneda embedding $Y_{\mathcal{C}} : \mathcal{C} \rightarrow \mathbf{Set}^{\mathcal{C}^{\text{op}}}$ is the generic model.

- If (\mathcal{C}, J) is a small site over a left exact category \mathcal{C} , then the theory

of left-exact (i.e. finite limit preserving) continuous (i.e. takes covering into colimits) functors is classified by the topos $\text{Sh}(\mathcal{C}, J)$, where the canonical model is $\mathcal{C} \xrightarrow{Y} \mathbf{Set}^{\mathcal{C}^{op}} \xrightarrow{a} \text{Sh}(\mathcal{C}, J)$.

- The Mitchell-Bénabou language of a elementary/Grothendieck topos and the Kripke-Joyal semantics allows us to interpret –in particular– first-order formulas in many sorted languages $L_{\omega\omega}/L_{\infty\omega}$ in a elementary/Grothendieck topos. Every geometric theory admits a classifying topos.

- Every Grothendieck topos is the classifying topos of a small geometric sketch.

2 A Classifying Topos for the Theory of \mathcal{C}^∞ –rings

In this section we describe a classifying topos for the theory of \mathcal{C}^∞ –rings. We mimic the construction of a classifying topos for the theory of commutative unital rings, given by I. Moerdijk and S. Mac Lane in [12], making some necessary adaptations to the context of \mathcal{C}^∞ –rings.

2.1 \mathcal{C}^∞ –Ring Objects in Categories with Finite Products

Definition 2.1 *Let \mathcal{C} be a category with finite products. A \mathcal{C}^∞ –ring object in \mathcal{C} is a morphism of sketches $A : \mathcal{S}_{\mathcal{C}^\infty\text{-Rng}} \rightarrow \text{sk}(\mathcal{C})$, where $\mathcal{S}_{\mathcal{C}^\infty\text{-Rng}}$ is the sketch of the theory of \mathcal{C}^∞ –rings.*

Proposition 2.2 *Given a \mathcal{C}^∞ –ring-object $A : \mathcal{S}_{\mathcal{C}^\infty\text{-Rng}} \rightarrow \text{sk}(\mathcal{C})$ in \mathcal{C} , in the sense of the **Definition 2.1**, the object $A(|\mathbb{R}|) \in \text{Obj}(\mathcal{C})$ has an obvious \mathcal{C}^∞ –ring structure, Ψ , given by:*

$$\begin{aligned} \Psi : \bigcup_{n \in \mathbb{N}} \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) &\rightarrow \bigcup_{n \in \mathbb{N}} \text{Hom}_{\mathcal{C}}(A(|\mathbb{R}|)^n, A(|\mathbb{R}|)) \\ f &\mapsto A(|f|) : A(|\mathbb{R}|)^n \rightarrow A(|\mathbb{R}|) \end{aligned}$$

Thus, we have the (universal-algebraic) \mathcal{C}^∞ –ring $(A(|\mathbb{R}|), \Psi)$.

Proof. It suffices to prove that Ψ satisfies the two groups of axioms given in **Definition 1.2**.

Ψ preserves projections, since A , as a \mathcal{C}^∞ –ring object, maps the projective cones given in \mathcal{P} to limit cones in \mathcal{C} – that is, to products. Given $n, m_1, \dots, m_k \in \mathbb{N}$ such that $n = m_1 + \dots + m_k$ and the projections $p_{m_i}^n : \mathbb{R}^n \rightarrow \mathbb{R}^{m_i}, i = 1, \dots, k$, $\Psi(p_{m_i}^n) := A(|p_{m_i}^n|) : A(|\mathbb{R}|)^n \rightarrow A(|\mathbb{R}|)^{m_i}$, which must be the projections since A maps the cone $(p_{m_i}^n : \mathbb{R}^n \rightarrow \mathbb{R}^{m_i})_{i=1, \dots, k}$ to a product in \mathcal{C} .

Also, for every $n \in \mathbb{N}$ and every $(n+2)$ -tuple of \mathcal{C}^∞ -functions, (h, g_1, \dots, g_n, f) with $f \in \mathcal{C}^\infty(\mathbb{R}^n)$, $g_1, \dots, g_n \in \mathcal{C}^\infty(\mathbb{R}^k)$ with:

$$h = f \circ (g_1, \dots, g_n)$$

we have:

$$\Psi(f \circ (g_1, \dots, g_n)) = A(|f| \circ (|g_1|, \dots, |g_n|)) = A(|f|) \circ A((|g_1|, \dots, |g_n|)),$$

since A , as a \mathcal{C}^∞ -ring, maps the diagram:

$$\begin{array}{ccc} |\mathbb{R}^k| & \xrightarrow{(|g_1|, \dots, |g_n|)} & |\mathbb{R}^n| \\ & \searrow |h| & \downarrow |f| \\ & & |\mathbb{R}| \end{array}$$

(that belongs to \mathcal{D} since $h = f \circ (g_1, \dots, g_n)$) to a commutative one:

$$\begin{array}{ccc} A(|\mathbb{R}^k|) & \xrightarrow{A((|g_1|, \dots, |g_n|))} & A(|\mathbb{R}^n|) \\ & \searrow A(|h|) & \downarrow A(|f|) \\ & & A(|\mathbb{R}|) \end{array}$$

that is $A(|h|) = A(|f|) \circ A((|g_1|, \dots, |g_n|))$.

Claim: $A((|g_1|, \dots, |g_n|)) = (A(|g_1|), \dots, A(|g_n|))$.

Indeed, for every $i \in \{1, \dots, k\}$ the following diagram commutes:

$$\begin{array}{ccc} A(|\mathbb{R}^k|) & \xrightarrow{A((|g_1|, \dots, |g_n|))} & A(|\mathbb{R}^n|) \\ & \searrow A(|g_i|) & \downarrow A(|p_i^n|) \\ & & A(|\mathbb{R}|) \end{array}$$

and since A interprets each p_i^n , $i = 1, \dots, k$, as a projection, $A(|p_i^n|)$, it follows that:

$$A((|g_1|, \dots, |g_n|)) = (A(|g_1|), \dots, A(|g_n|)).$$

Thus

$$\begin{aligned} \Psi(h) &:= A(|h|) = A(|f| \circ (|g_1|, \dots, |g_n|)) = A(|f|) \circ A((|g_1|, \dots, |g_n|)) = \\ &= A(|f|) \circ (A(|g_1|), \dots, A(|g_n|)) = \Psi(f) \circ (\Psi(g_1), \dots, \Psi(g_n)) \end{aligned}$$

and Ψ is a \mathcal{C}^∞ -ring structure. ■

Remark 2.3 *Let \mathcal{C} be a category with all finite limits. The category $\underline{\mathcal{C}^\infty - \text{Ring}}(\mathcal{C})$ is not a subcategory of \mathcal{C} (cf. p. 101 of [17]). However, there is a forgetful functor $U : \underline{\mathcal{C}^\infty - \text{Ring}}(\mathcal{C}) \rightarrow \mathcal{C}$ which is faithful and reflects isomorphisms (cf. **Proposition 11.3.3** of [17]). It follows that U reflects all the limits and colimits that it preserves and which exist in $\underline{\mathcal{C}^\infty - \text{Ring}}(\mathcal{C})$.*

The following proposition gives us some properties of the category $\underline{\mathcal{C}^\infty - \text{Ring}}(\mathcal{C})$ which are inherited from \mathcal{C} .

Proposition 2.4 *If a category \mathcal{C} is finitely complete, then the same is true for the category $\underline{\mathcal{C}^\infty - \text{Ring}}(\mathcal{C})$.*

Proof. It is an immediate application of **Proposition 11.5.1** of page 103 of [17]. ■

Proposition 2.5 *Let \mathcal{C} be a category with all finite limits. Every left-exact functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ induces a functor:*

$$T_{\underline{\mathcal{C}^\infty - \text{Ring}}} : \underline{\mathcal{C}^\infty - \text{Ring}}(\mathcal{C}) \rightarrow \underline{\mathcal{C}^\infty - \text{Ring}}(\mathcal{C}')$$

Proof. Since every functor preserves commutative diagrams, it follows that F maps commutative diagrams of \mathcal{C} to commutative diagrams of \mathcal{C}' , so the \mathcal{C}^∞ -ring-objects of \mathcal{C} are mapped to \mathcal{C}^∞ -ring-objects of \mathcal{C}' . ■

Proposition 2.6 *The object $\mathcal{C}^\infty(\mathbb{R})$ of $\underline{\mathcal{C}^\infty \mathbf{Rng}}_{\text{fp}}$ is a \mathcal{C}^∞ -ring-object in*

$$\underline{\mathcal{C}^\infty \mathbf{Rng}}_{\text{fp}},$$

Proof. Given any $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \subseteq \bigcup_{n \geq 0} \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ we define \widehat{f} as the unique \mathcal{C}^∞ -homomorphism sending the identity function $\text{id}_{\mathbb{R}} : \mathbb{R} \rightarrow \mathbb{R}$ to f , that is:

$$\begin{aligned} \widehat{f} = f \circ - : \mathcal{C}^\infty(\mathbb{R}) &\rightarrow \mathcal{C}^\infty(\mathbb{R}^n) \\ g &\mapsto f \circ g \end{aligned}$$

■

Theorem 2.7 *The category*

$$\mathcal{C}^\infty \mathbf{Rng}_{\text{fp}}^{\text{op}}$$

is a category with finite limits freely generated by the \mathcal{C}^∞ -ring-object $\mathcal{C}^\infty(\mathbb{R})$.

Proof. As we have already commented, this amounts to prove that for any category with finite limits, \mathcal{C} , the evaluation of a left-exact functor $F : \mathcal{C}^\infty \mathbf{Rng}_{\text{fp}}^{\text{op}} \rightarrow \mathcal{C}$ at $\mathcal{C}^\infty(\mathbb{R})$ gives the following equivalence of categories:

$$\begin{array}{ccc} \text{ev}_{\mathcal{C}^\infty(\mathbb{R})} : \mathbf{Lex}(\mathcal{C}^\infty \mathbf{Rng}_{\text{fp}}^{\text{op}}, \mathcal{C}) & \rightarrow & \underline{\mathcal{C}^\infty - \text{Rings}}(\mathcal{C}) \\ & F \mapsto & F(\mathcal{C}^\infty(\mathbb{R})) \end{array}$$

First note that this correspondence is indeed a function, for if F is left-exact, then it preserves \mathcal{C}^∞ -ring-objects, hence it sends the \mathcal{C}^∞ -ring object $\mathcal{C}^\infty(\mathbb{R})$ of $\mathcal{C}^\infty \mathbf{Rng}_{\text{fp}}^{\text{op}}$ into a \mathcal{C}^∞ -ring object of \mathcal{C} .

We are going to show that this functor is full, faithful and dense.

- $\text{ev}_{\mathcal{C}^\infty(\mathbb{R})}$ **is faithful;**

Let $F, G \in \text{Obj}(\mathbf{Lex}(\mathcal{C}^\infty \mathbf{Rng}_{\text{fp}}^{\text{op}}, \mathcal{C}))$ and let $\eta, \theta : F \Rightarrow G$ be two natural transformations between them such that:

$$(\eta_{\mathcal{C}^\infty(\mathbb{R})} : F(\mathcal{C}^\infty(\mathbb{R})) \rightarrow G(\mathcal{C}^\infty(\mathbb{R}))) = (\theta_{\mathcal{C}^\infty(\mathbb{R})} : F(\mathcal{C}^\infty(\mathbb{R})) \rightarrow G(\mathcal{C}^\infty(\mathbb{R}))).$$

We prove that given any object A of $\mathcal{C}^\infty - \mathbf{Rng}_{\text{fp}}^{\text{op}}$, we have $\eta_A = \theta_A$.

First suppose $A = \mathcal{C}^\infty(\mathbb{R}^n)$, that is, $A = \mathcal{C}^\infty(\mathbb{R}) \otimes_\infty \dots \otimes_\infty \mathcal{C}^\infty(\mathbb{R})$ (which is a product in $\mathcal{C}^\infty - \mathbf{Rng}_{\text{fp}}^{\text{op}}$). Since F is left-exact, $F(\mathcal{C}^\infty(\mathbb{R}^n)) = F(\mathcal{C}^\infty(\mathbb{R}))^n$, and:

$$\eta_{\mathcal{C}^\infty(\mathbb{R}^n)} = \eta_{\mathcal{C}^\infty(\mathbb{R})} \times \dots \times \eta_{\mathcal{C}^\infty(\mathbb{R})} : F(\mathcal{C}^\infty(\mathbb{R}))^n \rightarrow G(\mathcal{C}^\infty(\mathbb{R}))^n$$

Since $\eta_{\mathcal{C}^\infty(\mathbb{R})} = \theta_{\mathcal{C}^\infty(\mathbb{R})}$, it follows that $\eta_{\mathcal{C}^\infty(\mathbb{R}^n)} = \theta_{\mathcal{C}^\infty(\mathbb{R}^n)}$.

- $\text{ev}_{\mathcal{C}^\infty(\mathbb{R})}$ **is full;**

Let $F, G \in \text{Obj}(\mathbf{Lex}(\mathcal{C}^\infty - \mathbf{Rng}_{\text{fp}}^{\text{op}}, \mathcal{C}))$ and let $\varphi : F(\mathcal{C}^\infty(\mathbb{R})) \rightarrow G(\mathcal{C}^\infty(\mathbb{R}))$ be a morphism in $\underline{\mathcal{C}^\infty - \text{Rings}}(\mathcal{C})$. It suffices to take $\eta : F \Rightarrow G$ such that $\eta_{\mathcal{C}^\infty(\mathbb{R})} = \varphi$.

- $\text{ev}_{\mathcal{C}^\infty(\mathbb{R})}$ **is isomorphism dense;**

Let R be any object in $\underline{\mathcal{C}^\infty - \mathbf{Rings}}(\mathcal{C})$.

Given this object R , we are going to construct $\phi_R \in \text{Obj}(\mathbf{Lex}(\mathcal{C}^\infty - \mathbf{Rng}_{\text{fp}}^{\text{op}}, \mathcal{C}))$ such that $\text{ev}_{\mathcal{C}^\infty(\mathbb{R})}(\phi_R) \cong R$.

We set $\phi_R(\mathcal{C}^\infty(\mathbb{R})) = R$.

We first define the action of ϕ_R on the free \mathcal{C}^∞ -ring objects.

Now, given a free \mathcal{C}^∞ -ring object on n generators, R^n , since ϕ_R is to be left-exact, it transforms coproducts in $\mathcal{C}^\infty - \mathbf{Rng}_{\text{fp}}$ into products of \mathcal{C} . Hence, since $\mathcal{C}^\infty(\mathbb{R}^n) \cong \mathcal{C}^\infty(\mathbb{R}) \otimes_\infty \cdots \otimes_\infty \mathcal{C}^\infty(\mathbb{R})$, we set:

$$\phi_R(\mathcal{C}^\infty(\mathbb{R}^n)) = R^n,$$

which establishes the action of ϕ_R on the free objects of $\mathcal{C}^\infty - \mathbf{Rng}_{\text{fp}}^{\text{op}}$.

Now we shall describe the action of ϕ_R on the arrows between objects of $\mathcal{C}^\infty - \mathbf{Rng}_{\text{fp}}^{\text{op}}$:

$$(\phi_R)_1 : \text{Mor}(\underline{\mathcal{C}^\infty - \mathbf{Rngs}}(\mathcal{C})) \rightarrow \text{Nat}(\mathbf{Lex}(\mathcal{C}^\infty - \mathbf{Rng}_{\text{fp}}^{\text{op}}, \mathcal{C}))$$

beginning with the \mathcal{C}^∞ -homomorphisms between the free objects of $\mathcal{C}^\infty - \mathbf{Rng}_{\text{fp}}^{\text{op}}$.

An arrow (i.e., a \mathcal{C}^∞ -homomorphism) in $\mathcal{C}^\infty - \mathbf{Rng}_{\text{fp}}$ between free \mathcal{C}^∞ -rings is a map:

$$\begin{aligned} p : \quad \mathcal{C}^\infty(\mathbb{R}^k) &\rightarrow \mathcal{C}^\infty(\mathbb{R}^n) \\ (\mathbb{R}^k \xrightarrow{g} \mathbb{R}) &\mapsto (\mathbb{R}^n \xrightarrow{p(g)} \mathbb{R}) \end{aligned}$$

given by a k -tuple of smooth functions, $(p_1, \dots, p_k) : \mathbb{R}^n \rightarrow \mathbb{R}^k$:

$$\begin{array}{ccc} \mathbb{R}^k & \xleftarrow{(p_1, \dots, p_k)} & \mathbb{R}^n \\ & \searrow g & \swarrow p(g) \\ & \mathbb{R} & \end{array}$$

where $p_i = p(\pi_i) : \mathbb{R}^n \rightarrow \mathbb{R}, i = 1, \dots, k$ and $\pi_i : \mathbb{R}^k \rightarrow \mathbb{R}$ is the projection on the i -th coordinate.

Each such smooth function $p_i : \mathbb{R}^n \rightarrow \mathbb{R}$ yields an arrow in \mathcal{C} :

$$p_i^{(R)} : R^n \rightarrow R \tag{1}$$

defined from the \mathcal{C}^∞ -ring structure (defined in the **Proposition 2.2**), say Ψ , of $R \in \underline{\mathcal{C}^\infty - \mathbf{Rings}}(\mathcal{C})$, which interprets every smooth function in \mathcal{C} .

We have, as a direct consequence of the fact pointed out by Moerdijk and Reyes on the page 21 of [16], a 1–1 correspondence between \mathcal{C}^∞ –homomorphisms from $\mathcal{C}^\infty(\mathbb{R}^k)$ to $\mathcal{C}^\infty(\mathbb{R}^n)$ and k –tuples of smooth functions from \mathbb{R}^n to \mathbb{R} :

$$\frac{p : \mathcal{C}^\infty(\mathbb{R}^k) \rightarrow \mathcal{C}^\infty(\mathbb{R}^n)}{\mathbb{R}^n \xrightarrow{\langle p_1, \dots, p_k \rangle} \mathbb{R}^k}.$$

The image under ϕ_R of the arrow $p : \mathcal{C}^\infty(\mathbb{R}^k) \rightarrow \mathcal{C}^\infty(\mathbb{R}^n)$ is calculated first taking the k -tuple of smooth functions given by the correspondence:

$$\frac{p : \mathcal{C}^\infty(\mathbb{R}^k) \rightarrow \mathcal{C}^\infty(\mathbb{R}^n)}{\mathbb{R}^n \xrightarrow{\langle p_1, \dots, p_k \rangle} \mathbb{R}^k}.$$

and then interpreting it in R :

$$\phi_R(\mathcal{C}^\infty(\mathbb{R}^k) \xrightarrow{p} \mathcal{C}^\infty(\mathbb{R}^n)) = p^{(R)} = (p_1^{(R)}, \dots, p_k^{(R)}) : R^n \rightarrow R^k \quad (2)$$

To complete the definition of the functor ϕ_R on any finitely presented \mathcal{C}^∞ –ring $\frac{\mathcal{C}^\infty(\mathbb{R}^n)}{\langle p_1, \dots, p_k \rangle}$, we note that, by definition, this quotient fits into a coequalizer diagram:

$$\mathcal{C}^\infty(\mathbb{R}^k) \xrightarrow[p]{\quad} \mathcal{C}^\infty(\mathbb{R}^n) \xrightarrow[\langle p_1, \dots, p_k \rangle]{q_{\langle p_1, \dots, p_k \rangle}} \frac{\mathcal{C}^\infty(\mathbb{R}^n)}{\langle p_1, \dots, p_k \rangle} \quad (3)$$

where $p_i = p(\pi_i)$ for $i = 1, \dots, k$ and $0(\pi_i) = 0$ for $i = 1, \dots, k$.

The category \mathcal{C} , by hypothesis, has all finite limits, so the category of the \mathcal{C}^∞ –rings objects in a category \mathcal{C} has equalizers, and there is an equalizer diagram:

$$E \xrightarrow{e} R^n \xrightarrow[0^{(R)}]{p^{(R)}} R^m$$

Thus we define the image under the contravariant functor ϕ_R of the finitely presented \mathcal{C}^∞ –ring $\frac{\mathcal{C}^\infty(\mathbb{R}^n)}{\langle p_1, \dots, p_k \rangle}$ as:

$$\phi_R\left(\frac{\mathcal{C}^\infty(\mathbb{R}^n)}{\langle p_1, \dots, p_k \rangle}\right) := E$$

that is, by the following equalizer diagram in \mathcal{C} :

$$\phi_R\left(\frac{\mathcal{C}^\infty(\mathbb{R}^n)}{\langle p_1, \dots, p_k \rangle}\right) \longrightarrow R^n \xrightarrow[0^{(R)}]{p^{(R)}} R^k \quad (4)$$

Next, we define ϕ_R on a \mathcal{C}^∞ -homomorphism $h : B \rightarrow C$ between any two finitely presented \mathcal{C}^∞ -rings. Let $\frac{\mathcal{C}^\infty(\mathbb{R}^n)}{\langle p_1, \dots, p_k \rangle}$ and $\frac{\mathcal{C}^\infty(\mathbb{R}^m)}{\langle g_1, \dots, g_t \rangle}$ be two finitely presented \mathcal{C}^∞ -rings and let:

$$\frac{\mathcal{C}^\infty(\mathbb{R}^n)}{\langle f_1, \dots, f_k \rangle} \xrightarrow{\Phi} \frac{\mathcal{C}^\infty(\mathbb{R}^m)}{\langle g_1, \dots, g_t \rangle}$$

be a \mathcal{C}^∞ -homomorphism. The \mathcal{C}^∞ -homomorphism Φ is determined by some \mathcal{C}^∞ -function:

$$\begin{aligned} \varphi : \mathbb{R}^m &\rightarrow \mathbb{R}^n \\ x &\mapsto (\varphi_1(x), \dots, \varphi_n(x)) \end{aligned}$$

such that $\langle f_1, \dots, f_k \rangle \subseteq \varphi_*[\langle g_1, \dots, g_t \rangle]$. Hence, the \mathcal{C}^∞ -homomorphism Φ is determined by the equivalence classes of n \mathcal{C}^∞ -functions: $\varphi_1, \dots, \varphi_n : \mathbb{R}^m \rightarrow \mathbb{R}$ such that:

$$(\forall j \in \{1, \dots, k\})(f_j \circ \varphi = f_j \circ (\varphi_1, \dots, \varphi_n) \in \langle g_1, \dots, g_t \rangle). \tag{5}$$

As in (2), these n smooth functions determine a \mathcal{C}^∞ -homomorphism $\varphi^{(R)} : R^m \rightarrow R^n$. Now $\phi_R \left(\frac{\mathcal{C}^\infty(\mathbb{R}^n)}{\langle f_1, \dots, f_k \rangle} \right)$ and $\phi_R \left(\frac{\mathcal{C}^\infty(\mathbb{R}^m)}{\langle g_1, \dots, g_t \rangle} \right)$ fit into equalizer rows:

$$\begin{array}{ccc} \phi_R \left(\frac{\mathcal{C}^\infty(\mathbb{R}^n)}{\langle f_1, \dots, f_k \rangle} \right) & \longrightarrow & R^n \xrightarrow[\cong_{0^{(R)}}]{f^{(R)}} R^k \\ & & \uparrow \varphi^{(R)} \\ \phi_R \left(\frac{\mathcal{C}^\infty(\mathbb{R}^m)}{\langle g_1, \dots, g_t \rangle} \right) & \xrightarrow{\alpha} & R^m \xrightarrow[\cong_{0^{(R)}}]{g^{(R)}} R^t \end{array}$$

where $f^{(R)} : R^n \rightarrow R^k$ is the interpretation of $f = (f_1, \dots, f_k) : \mathbb{R}^n \rightarrow \mathbb{R}^k$, $g^{(R)} : R^m \rightarrow R^t$ is the interpretation of $g = (g_1, \dots, g_t) : \mathbb{R}^m \rightarrow \mathbb{R}^t$, the equalizer α in the lower left is determined by m arrows, $\alpha = (\alpha_1, \dots, \alpha_m)$ with

$$\alpha_s : \phi_R \left(\frac{\mathcal{C}^\infty(\mathbb{R}^m)}{\langle g_1, \dots, g_t \rangle} \right) \rightarrow R, s = 1, \dots, m$$

which satisfy (by definition) $g_\ell(\alpha_1, \dots, \alpha_m) = 0$ for every $\ell \in \{1, \dots, t\}$.

We have:

$$f \circ \varphi = (f_1 \circ \varphi, \dots, f_k \circ \varphi)$$

so

$$f^{(R)} \circ \varphi^{(R)} = (f_1^{(R)} \circ \varphi^{(R)}, \dots, f_k^{(R)} \circ \varphi^{(R)}).$$

Since for every $i = 1, \dots, k$, $f_i \circ \varphi \in \langle g_1, \dots, g_t \rangle$, there are ℓ $\mu_1, \dots, \mu_t \in \mathcal{C}^\infty(\mathbb{R}^m)$ such that:

$$f_i^{(R)} \circ \varphi^{(R)} = \sum_{\ell=1}^t \mu_\ell \cdot g_\ell,$$

and by (5), it follows that:

$$(\forall i \in \{1, \dots, k\})(f_i^{(R)} \circ \varphi^{(R)})(\alpha_1, \dots, \alpha_m) = \sum_{\ell=1}^t \mu_\ell(\alpha_1, \dots, \alpha_m) \cdot \underbrace{g_\ell(\alpha_1, \dots, \alpha_m)}_{=0},$$

so

$$f^{(R)} \circ (\varphi^{(R)} \circ \alpha) = (f_1^{(R)} \circ \varphi^{(R)}, \dots, f_k^{(R)} \circ \varphi^{(R)}) = 0^{(R)}.$$

Hence, the composite $\varphi^{(R)} \circ \alpha$ consists of n arrows to R which satisfy the conditions $f \circ (\varphi^{(R)} \circ \alpha) = 0$.

Therefore, by the universal property of equalizers, there is a unique arrow $\phi_R(h)$, indicated as follows:

$$\begin{array}{ccc} \phi_R(B) & \xrightarrow{\quad} & R^n \xrightleftharpoons[0^{(R)}]{f^{(R)}} R^k \\ \uparrow \exists! \phi_R(\Phi) \downarrow & \nearrow \varphi^{(R)} \circ \alpha & \\ \phi_R(C) & & \end{array}$$

Note that $\phi_R(\Phi)$ is independent of the choice of φ_i in their equivalence classes, so ϕ_R is a functor, as required in (4).

Claim: For each \mathcal{C}^∞ -ring object R in \mathcal{C} , the functor ϕ_R thus defined is a left-exact functor $\phi_R : \mathcal{C}^\infty \mathbf{Rng}_{\text{fp}} \rightarrow \mathcal{C}$.

We are going to show that ϕ_R preserves terminal object, binary products and equalizers, so ϕ_R will preserve all finite limits (which are constructed from these).

In fact, $\phi_R(\mathbb{R}^0)$ is the empty product of copies of R [since $\phi_R(\mathbb{R}^0) = R^0$ for $n = 0$], i.e., $\phi_R(\mathbb{R}^0) = 1$, so ϕ_R preserves the terminal object.

Also, since the product of two equalizer diagrams is again an equalizer, one easily verifies from (4) that ϕ_R is such that for any $\frac{\mathcal{C}^\infty(\mathbb{R}^n)}{\langle f_1, \dots, f_k \rangle}$ and any $\frac{\mathcal{C}^\infty(\mathbb{R}^m)}{\langle g_1, \dots, g_t \rangle}$ we have:

$$\phi_R \left(\frac{\mathcal{C}^\infty(\mathbb{R}^n)}{\langle f_1, \dots, f_k \rangle} \otimes_\infty \frac{\mathcal{C}^\infty(\mathbb{R}^m)}{\langle g_1, \dots, g_t \rangle} \right) \cong \phi_R \left(\frac{\mathcal{C}^\infty(\mathbb{R}^n)}{\langle f_1, \dots, f_k \rangle} \right) \times \phi_R \left(\frac{\mathcal{C}^\infty(\mathbb{R}^m)}{\langle g_1, \dots, g_t \rangle} \right)$$

that is, ϕ_R preserves binary products.

Finally, to see that ϕ_R preserves equalizers, consider a coequalizer constructed in the evident way from two arbitrary maps s, s' in the category of finitely presented \mathcal{C}^∞ -rings,

$$\begin{array}{ccc} \frac{\mathcal{C}^\infty(\mathbb{R}^m)}{\langle p_1, \dots, p_k \rangle} & \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{s'} \end{array} & \frac{\mathcal{C}^\infty(\mathbb{R}^n)}{\langle g_1, \dots, g_t \rangle} \longrightarrow \twoheadrightarrow \\ & \twoheadrightarrow & \frac{\mathcal{C}^\infty(\mathbb{R}^n)}{\langle g_1, \dots, g_t, s \circ \pi_1 - s' \circ \pi_1, \dots, s \circ \pi_k - s' \circ \pi_k \rangle} \end{array} \quad (6)$$

We must show that ϕ_R sends this coequalizer (6) to an equalizer diagram in \mathcal{C} .

First of all, if (6) is a coequalizer, then so is the diagram:

$$\begin{array}{ccc} \mathcal{C}^\infty(\mathbb{R}^m) & \begin{array}{c} \xrightarrow{s \circ q_I} \\ \xrightarrow{s' \circ q_I} \end{array} & \frac{\mathcal{C}^\infty(\mathbb{R}^n)}{\langle g_1, \dots, g_t \rangle} \longrightarrow \twoheadrightarrow \\ & \twoheadrightarrow & \frac{\mathcal{C}^\infty(\mathbb{R}^n)}{\langle g_1, \dots, g_t, s \circ \pi_1 \circ q_I - s' \circ \pi_1 \circ q_I, \dots, s \circ \pi_k \circ q_I - s' \circ \pi_k \circ q_I \rangle} \end{array} \quad (7)$$

obtained by precomposing (6) with the epimorphism $q_I : \mathcal{C}^\infty(\mathbb{R}^m) \rightarrow \frac{\mathcal{C}^\infty(\mathbb{R}^m)}{\langle p_1, \dots, p_k \rangle}$.

Moreover, since ϕ_R sends the latter epimorphism, q_I , to a monomorphism in \mathcal{C} [in fact, to an equalizer, as in (4), and every equalizer is a monomorphism], ϕ_R sends (6) to an equalizer if, and only if it does for (7). So it suffices to show that ϕ_R sends coequalizers of the special form (7) to equalizers in \mathcal{C} .

Next, since (7) is a coequalizer, so is

$$\begin{array}{ccc} \mathcal{C}^\infty(\mathbb{R}^m) & \begin{array}{c} \xrightarrow{s \circ q_I - s' \circ q_I} \\ \xrightarrow{s' \circ q_I} \end{array} & \frac{\mathcal{C}^\infty(\mathbb{R}^n)}{\langle g_1, \dots, g_t \rangle} \longrightarrow \twoheadrightarrow \\ & \twoheadrightarrow & \frac{\mathcal{C}^\infty(\mathbb{R}^n)}{\langle g_1, \dots, g_t, s \circ \pi_1 \circ q_I - s' \circ \pi_1 \circ q_I, \dots, s \circ \pi_k \circ q_I - s' \circ \pi_k \circ q_I \rangle} \end{array} \quad (8)$$

and one readily checks that ϕ_R sends (7) to an equalizer in \mathcal{C} if, and only if it does for (8). So, by replacing s by $s - s'$ and s' by 0 in (7) we see that it suffices

to show that ϕ_R sends coequalizers of the form (7) with $s' = 0$ to equalizers in \mathcal{C} .

Given a \mathcal{C}^∞ -homomorphism $p : \mathcal{C}^\infty(\mathbb{R}^k) \rightarrow \mathcal{C}^\infty(\mathbb{R}^n)$, construct the diagram:

$$\begin{array}{ccccc}
 & & \mathcal{C}^\infty(\mathbb{R}^k) & & \\
 & & \downarrow p \parallel 0 & & \\
 \mathcal{C}^\infty(\mathbb{R}^{m+k}) & \xrightarrow[\parallel 0]{(s,p)} & \mathcal{C}^\infty(\mathbb{R}^n) & \xrightarrow{\quad} & \frac{\mathcal{C}^\infty(\mathbb{R}^n)}{\langle p_1, \dots, p_k, s \circ \pi_1, \dots, s \circ \pi_m \rangle} \\
 \downarrow \pi_i \mapsto 0 \atop m+1 \leq i \leq k & & \downarrow & & \parallel \\
 \mathcal{C}^\infty(\mathbb{R}^m) & \xrightarrow[\parallel 0]{s} & \mathcal{C}^\infty(\mathbb{R}^n) & \xrightarrow{\quad} & \frac{\mathcal{C}^\infty(\mathbb{R}^n)}{\langle p_1, \dots, p_k, s \circ \pi_1, \dots, s \circ \pi_m \rangle}
 \end{array}$$

consisting of three coequalizers, two of the form (3). By definition (4), ϕ_R sends both the vertical coequalizer and the upper horizontal coequalizer to equalizers in \mathcal{C} . It follows, by diagram chasing that it also sends the lower horizontal coequalizer to an equalizer in \mathcal{C} .

This shows that ϕ_R is a left-exact functor.

By construction, $\text{ev}_{\mathcal{C}^\infty(\mathbb{R})}(\phi_R) = \phi_R(\mathcal{C}^\infty(\mathbb{R})) = R$, so $\text{ev}_{\mathcal{C}^\infty(\mathbb{R})}$ is a fully faithful dense functor, hence an equivalence of categories. ■

We can, therefore, summarize the result of this section as follows:

Theorem 2.8 *The presheaf topos $\mathbf{Sets}^{\mathcal{C}^\infty\mathbf{Rng}_{\text{fp}}}$ is a classifying topos for \mathcal{C}^∞ -rings, and the universal \mathcal{C}^∞ -ring R is the \mathcal{C}^∞ -ring object in $\mathbf{Sets}^{\mathcal{C}^\infty\mathbf{Rng}_{\text{fp}}}$ given by the forgetful functor from $\mathcal{C}^\infty\mathbf{Rng}_{\text{fp}}$ to \mathbf{Sets} . Thus, for any co-complete topos \mathcal{E} there is an equivalence of categories, natural in \mathcal{E} :*

$$\begin{array}{ccc}
 \text{Geom}(\mathcal{E}, \mathbf{Sets}^{\mathcal{C}^\infty\mathbf{Rng}_{\text{fp}}}) & \rightarrow & \mathcal{C}^\infty\mathbf{Rings}(\mathcal{E}) \\
 f & \mapsto & f^*(R)
 \end{array}$$

3 A Classifying Topos for the Theory of local \mathcal{C}^∞ -rings

Now we describe the \mathcal{C}^∞ -analog of the Zariski site, whose corresponding topos of sheaves will be the classifying topos of the theory of the \mathcal{C}^∞ -local rings.

3.1 The Smooth Zariski Site

In the following we describe the \mathcal{C}^∞ -analog of the Zariski site, which classifies the theory of the \mathcal{C}^∞ -local rings.

It is known that the topos of sheaves over the Zariski site classifies the theory of (commutative unital) local rings (see, for example, [13]). We briefly recall its construction.

Let \mathcal{C} be (some) skeleton of the category of all finitely presented commutative unital rings, $\mathbf{CRing}_{\text{fp}}$. Given a finitely presented commutative unital ring, A , we say that a finite family of ring homomorphisms, $\{f_i : A \rightarrow B_i \mid i \in \{1, \dots, n\}\}$ is a “co-coverage” of A if, and only if there are $a_1, \dots, a_n \in A$ with $\langle \{a_1, \dots, a_n\} \rangle = A$ such that for every $i \in \{1, \dots, n\}$, $(B_i, A \xrightarrow{f_i} B_i) \cong (A[a_i^{-1}], \eta_{a_i} : A \rightarrow A[a_i^{-1}])$. The set of all co-covering families of A is denoted by $\text{coCov}(A)$. Naturally, given any isomorphism $\varphi : A \rightarrow B$, $\{A \xrightarrow{\varphi} B\} \in \text{coCov}(A)$, and for any set of generators of A , $\{a_1, \dots, a_n\}$, $\{\eta_{a_i} : A \rightarrow A[a_i^{-1}] \mid i \in \{1, \dots, n\}\} \in \text{coCov}(A)$.

Passing to the opposite category, \mathcal{C}^{op} , we say that a finite set of arrows $\{f_i : B_i \rightarrow A \mid i \in \{1, \dots, n\}\}$ is a “covering family for A ” if, and only if $\{f_i^{\text{op}} : A \rightarrow B_i \mid i \in \{1, \dots, n\}\} \in \text{coCov}(A)$, and we write $\{f_i : B_i \rightarrow A \mid i \in \{1, \dots, n\}\} \in \text{Cov}(A)$. The Grothendieck-Zariski topology on \mathcal{C}^{op} is the one generated by Cov , J_{Cov} , that is, given any commutative unital ring A , $J_{\text{Cov}}(A)$ consists of all sieves S on A generated by $\text{Cov}(A)$, that is, $S \subseteq \cup_{C \in \text{Obj}(\mathcal{C})} \text{Hom}_{\mathcal{C}}(C, A)$ such that every $g \in S$ factors through some element of $\text{Cov}(A)$.

The pair $(\mathcal{C}^{\text{op}}, J_{\text{Cov}})$ thus obtained is the so-called “Zariski site”. The topos of sheaves over $(\mathcal{C}^{\text{op}}, J_{\text{Cov}})$, $\mathcal{Z} = \text{Sh}(\mathcal{C}^{\text{op}}, J_{\text{Cov}})$ is the classifying topos for the theory of local commutative unital rings.

In order to define the covering families for \mathcal{C}^∞ -rings we need, just as in the algebraic case, an appropriate notion of “a \mathcal{C}^∞ -ring of fractions”. Given a \mathcal{C}^∞ -ring A and an element $a \in A$, the \mathcal{C}^∞ -ring of fractions of A with respect to a must be a pair $(B, f : A \rightarrow B)$ such that $f(a) \in B^\times$, with the following universal property: given any \mathcal{C}^∞ -homomorphism $g : A \rightarrow C$ such that $g(a) \in C^\times$, there is a unique \mathcal{C}^∞ -homomorphism $\tilde{g} : B \rightarrow C$ such that the following diagram commutes:

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 & \searrow g & \downarrow \tilde{g} \\
 & & C
 \end{array}$$

In the **Theorem 1.4** of [14], I. Moerdijk and G. Reyes give two conditions which capture the notion of “the \mathcal{C}^∞ -ring of fractions with respect to one

element". The following definition we give can be shown to be its natural extension to arbitrary subsets.

Definition 3.1 *Let A be a \mathcal{C}^∞ -ring and let $S \subseteq A$. The \mathcal{C}^∞ -ring of fractions of A with respect to S is a pair $(A\{S^{-1}\}, \eta_S : A \rightarrow A\{S^{-1}\})$ where $A\{S^{-1}\}$ is a \mathcal{C}^∞ -ring and $\eta_S : A \rightarrow A\{S^{-1}\}$ is a \mathcal{C}^∞ -homomorphism such that:*

- (i) $(\forall \beta \in A\{S^{-1}\})(\exists c \in A)(\exists d \in A)((\eta_S(c) \in (A\{S^{-1}\})^\times) \& (\beta \cdot \eta_S(c) = \eta_S(d)))$;
- (ii) $(\forall a \in A)(\eta_S(a) = 0 \rightarrow (\exists c \in A)(\eta_S(c) \in (A\{S^{-1}\})^\times)(a \cdot c = 0))$.

Moreover, given any \mathcal{C}^∞ -ring B and any \mathcal{C}^∞ -homomorphism $\varphi : A \rightarrow B$ such that the pair $(B, \varphi : A \rightarrow B)$ satisfies (i) and (ii), then it can be shown that it is isomorphic to the \mathcal{C}^∞ -ring of fractions, $(A\{S^{-1}\}, \eta_S : A \rightarrow A\{S^{-1}\})$. Sometimes, when it must be stressed of which \mathcal{C}^∞ -ring we are inverting a subset, we write $\eta_S^A : A \rightarrow A\{S^{-1}\}$ instead of just $\eta_S : A \rightarrow A\{S^{-1}\}$.

We introduce the \mathcal{C}^∞ -analog of the (algebraic) concept of saturation of a multiplicative subset of a ring in the following:

Definition 3.2 *Let A be a \mathcal{C}^∞ -ring and let $S \subseteq A$. The \mathcal{C}^∞ -saturation of S is given by:*

$$S^{\infty\text{-sat}} = \eta_S^{-1}[A\{S^{-1}\}^\times]$$

where $A\{S^{-1}\}$ and $\eta_S : A \rightarrow A\{S^{-1}\}$ were given in **Definition 3.1**

Notation: In virtue of **Definition 3.1**, given any $\beta \in A\{S^{-1}\}$, there are $b \in A$ and $c \in S^{\infty\text{-sat}}$ such that $\beta \cdot \eta_S(c) = \eta_S(d)$, so we write $\beta = \frac{\eta_S(d)}{\eta_S(c)}$. For typographical reasons, whenever $S = \{a\} \subseteq A$, we also write A_a to denote $A\{a^{-1}\}$.

Combining these concepts, we are able to describe the co-covering families of the smooth Zariski Grothendieck (pre)topology.

Let \mathcal{C} be (some) skeleton of $\mathcal{C}^\infty\mathbf{Rng}_{\text{fp}}$. We first define the **smooth Grothendieck-Zariski pretopology** on \mathcal{C}^{op} .

Convention: We say that a covering family of A , $\{g_j : B_j \rightarrow A | j \in J\} \in \text{Cov}(A)$ (or a co-covering family of $\text{coCov}(A)$) **is generated by a family of \mathcal{C}^∞ -homomorphisms** $\mathcal{F} = \{f_i : A_i \rightarrow A | i \in I\}$ if, and only if $\{g_j : B_j \rightarrow A | j \in J\}$ consists of all the \mathcal{C}^∞ -homomorphism with codomain A which are isomorphic (in the comma category $\mathcal{C}^\infty\mathbf{Rng}_{\text{fp}} \downarrow A$) to some element of \mathcal{F} . We shall denote it by:

$$\{g_j : B_j \rightarrow A | j \in J\} \doteq \langle \{f_i : A_i \rightarrow A | i \in I\} \rangle = \langle \mathcal{F} \rangle$$

The covering families, in our case, will be “generated” by the dual (opposite) of the co-covering families defined as follows:

Let:

$$\begin{aligned} \text{coCov} : \text{Obj}(\mathcal{C}^\infty \mathbf{Rng}_{\text{fp}}) &\rightarrow \wp(\wp(\text{Mor}(\mathcal{C}^\infty \mathbf{Rng}_{\text{fp}}))) \\ A &\mapsto \text{coCov}(A) \end{aligned}$$

For every n -tuple of elements of A , $(a_1, \dots, a_n) \in A \times A \times \dots \times A$, $n \in \mathbb{N}$, such that $\langle a_1, a_2, \dots, a_n \rangle = A$, a family of \mathcal{C}^∞ -homomorphisms $k_i : A \rightarrow B_i$ such that:

- (i) For every $i \in \{1, \dots, n\}$, $k_i(a_i) \in B_i^\times$;
- (ii) For every $i \in \{1, \dots, n\}$, if $k_i(a) = 0$ for some $a \in A$, there is some $s_i \in \{a_i\}^{\infty\text{-sat}}$ such that $a \cdot s_i = 0$;
- (iii) For every $b \in B_i$ there are $c \in \{a_i\}^{\infty\text{-sat}}$ and $d \in A$ such that $b \cdot k_i(c) = k_i(d)$;

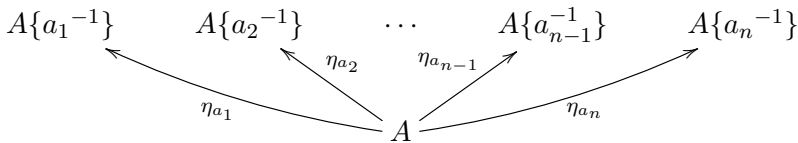
will be a co-covering family of the \mathcal{C}^∞ -ring A , that is:

$$\begin{aligned} \text{coCov}(A) &= \{ \mathcal{F} \subseteq \cup_{B \in \text{Obj}(\mathcal{C})} \text{Hom}_{\mathcal{C}^\infty \mathbf{Rng}_{\text{fp}}}(A, B) \mid \mathcal{F} = \\ &= \{ k_i : A \rightarrow B_i \mid (n \in \mathbb{N}) \& (i \in \{1, \dots, n\}) \& k_i \text{ satisfies (i), (ii) and (iii)} \} \end{aligned}$$

In other words,

$$\begin{aligned} \text{coCov}(A) &= \\ &= \{ \mathcal{F} \subseteq \cup_{B \in \text{Obj}(\mathcal{C})} \text{Hom}_{\mathcal{C}^\infty \mathbf{Rng}_{\text{fp}}}(A, B) \mid \mathcal{F} = \langle \eta_{a_i} : A \rightarrow A\{a_i^{-1}\} \mid i \in \{1, \dots, n\} \rangle \} \end{aligned}$$

In terms of diagrams, the “generators” of the co-covering families are given by the following arrows:



Given a finitely presented \mathcal{C}^∞ -ring, a **covering family** for A in $\mathcal{C}^\infty \mathbf{Rng}_{\text{fp}}^{\text{op}}$ is given by:

$$\text{Cov}(A) = \{ f^{\text{op}} : B \rightarrow A \mid (f : A \rightarrow B) \in \text{coCov}(A) \}$$

The following technical result is used in the sequel:

Proposition 3.3 *Let A be a \mathcal{C}^∞ -ring and let $a \in A$ and $\beta \in A\{a^{-1}\}$. Since $\beta = \eta_a^A(b)/\eta_a^A(c)$ for some $b \in A$ and $c \in \{a\}^{\infty\text{-sat}}$, then there is a unique \mathcal{C}^∞ -isomorphism of A -algebras:*

$$\theta_{ab} : (A\{a^{-1}\})\{\beta^{-1}\} \xrightarrow{\cong} A\{(a \cdot b)^{-1}\}$$

I.e., $\theta_{ab} : (A\{a^{-1}\})\{\beta^{-1}\} \rightarrow A\{(a \cdot b)^{-1}\}$ is a \mathcal{C}^∞ -rings isomorphism such that the following diagram commutes:

$$\begin{array}{ccccc} A & \xrightarrow{\eta_a^A} & A\{a^{-1}\} & \xrightarrow{\eta_\beta^{A_a}} & (A\{a^{-1}\})\{\beta^{-1}\} \\ & \searrow & & & \downarrow \theta_{ab} \\ & & & & A\{(a \cdot b)^{-1}\} \\ & \searrow \eta_{a \cdot b}^A & & & \end{array}$$

that is, $(\eta_{a \cdot b}^A : A \rightarrow A\{(a \cdot b)^{-1}\}) \cong (\eta_\beta^{A_a} \circ \eta_a^A : A \rightarrow (A\{a^{-1}\})\{\beta^{-1}\})$ in $A \downarrow \mathcal{C}^\infty \mathbf{Rng}_{\text{fp}}$. Hence:

$$\langle \{\eta_{a \cdot b} : A \rightarrow A\{(a \cdot b)^{-1}\} \mid a, b \in A\} \rangle = \langle \{\eta_\beta \circ \eta_a : A \rightarrow (A\{a^{-1}\})\{\beta^{-1}\} \mid a, b \in A\} \rangle.$$

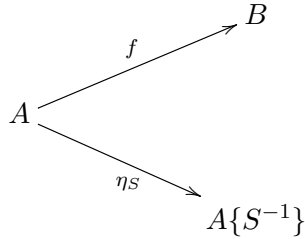
Proposition 3.4 *Let A and B be two \mathcal{C}^∞ -rings and $S \subseteq A$ and $f : A \rightarrow B$ a \mathcal{C}^∞ -homomorphism. By the universal property of $\eta_S : A \rightarrow A\{S^{-1}\}$ we have a unique \mathcal{C}^∞ -homomorphism $f_S : A\{S^{-1}\} \rightarrow B\{f[S]^{-1}\}$ such that the following square commutes:*

$$\begin{array}{ccc} A & \xrightarrow{\eta_S} & A\{S^{-1}\} \\ f \downarrow & & \downarrow \exists! f_S \\ B & \xrightarrow{\eta_{f[S]}} & B\{f[S]^{-1}\} \end{array}$$

The diagram:

$$\begin{array}{ccc} B & & \\ & \searrow \eta_{f[S]} & \\ & & B\{f[S]^{-1}\} \\ & \nearrow f_S & \\ A\{S^{-1}\} & & \end{array}$$

is a pushout of the diagram:



Remark 3.5 Note that if A is a finitely presented \mathcal{C}^∞ -ring, i.e. $A \cong \frac{\mathcal{C}^\infty(\mathbb{R}^n)}{\langle f_1, \dots, f_k \rangle}$, and $b \in A$, then $A\{b^{-1}\}$ is a finitely presented \mathcal{C}^∞ -ring:

$$A\{b^{-1}\} \cong \frac{A\{x\}}{(bx - 1)} \cong \frac{\mathcal{C}^\infty(\mathbb{R}^{n+1})}{\langle f_1 \circ \pi_1, \dots, f_k \circ \pi_1, (bx - 1) \circ \pi_2 \rangle}$$

Remark 3.6 Note that if A is a finitely presented \mathcal{C}^∞ -ring, i.e. $A \cong \frac{\mathcal{C}^\infty(\mathbb{R}^n)}{\langle f_1, \dots, f_k \rangle}$, and $b \in A$, then $A\{b^{-1}\}$ is a finitely presented \mathcal{C}^∞ -ring:

$$A\{b^{-1}\} \cong \frac{A\{x\}}{(bx - 1)} \cong \frac{\mathcal{C}^\infty(\mathbb{R}^{n+1})}{\langle f_1 \circ \pi_1, \dots, f_k \circ \pi_1, (bx - 1) \circ \pi_2 \rangle}$$

Definition 3.7 Let A be a \mathcal{C}^∞ -ring and let $I \subseteq A$ be an ideal. The \mathcal{C}^∞ -radical ideal of I is given by:

$$\sqrt[\infty]{I} = \left\{ a \in A \mid \left(\frac{A}{I} \right) \{a + I^{-1}\} \cong 0 \right\}$$

Definition 3.8 Given a \mathcal{C}^∞ -ring A , the smooth Zariski spectrum of A is given by the set:

$$\text{Spec}^\infty(A) = \{ \mathfrak{p} \in \text{Spec}(A) \mid \sqrt[\infty]{\mathfrak{p}} = \mathfrak{p} \}$$

together with the topology generated by the following sub-basic sets:

$$D^\infty(a) = \{ \mathfrak{p} \in \text{Spec}^\infty(A) \mid a \notin \mathfrak{p} \}$$

for each $a \in A$.

Proposition 3.9 Let A be a \mathcal{C}^∞ -ring. A family $\{a_1, \dots, a_n\} \subseteq A$ is such that $\langle \{a_1, \dots, a_n\} \rangle = A$ if, and only if:

$$\text{Spec}^\infty(A) = \bigcup_{i=1}^n D^\infty(a_i)$$

Proposition 3.10 *Cov is a Grothendieck pretopology on $\mathcal{C}^\infty\mathbf{Rng}_{\text{fp}}^{\text{op}}$.*

Proof. Let A be any finitely presented \mathcal{C}^∞ -ring.

Isomorphism axiom:

Whenever $\varphi^{\text{op}} : A' \rightarrow A$ is a \mathcal{C}^∞ -isomorphism, the family $\{\varphi^{\text{op}} : A' \rightarrow A\} \in \text{Cov}(A)$.

Note that $\varphi^{\text{op}} : A' \rightarrow A$ is a \mathcal{C}^∞ -isomorphism in $\mathcal{C}^\infty\mathbf{Rng}_{\text{fp}}^{\text{op}}$ if, and only if $\varphi : A \rightarrow A'$ is a \mathcal{C}^∞ -isomorphism in $\mathcal{C}^\infty\mathbf{Rng}_{\text{fp}}$. Thus, we are going to show that if $\varphi : A \rightarrow A'$ is a \mathcal{C}^∞ -isomorphism, then $\{\varphi : A \rightarrow A'\} \in \text{coCov}(A)$.

Indeed, $1_A \in A$ is such that $\langle 1_A \rangle = A$, so the one element family $\{\eta_{1_A} : A \rightarrow A\{1_A^{-1}\}\} \in \text{coCov}(A)$.

Since $\varphi : A \rightarrow A'$ is a \mathcal{C}^∞ -isomorphism, φ is, in particular, a \mathcal{C}^∞ -homomorphism, and we have, for every $s \in \{1_A\}^{\infty\text{-sat}} = A^\times$, $\varphi(s) \in A'^\times$. Also, if $\varphi(a) = 0_{A'}$ for some $a \in A$, since $\ker \varphi = \{0_A\}$ (for φ is injective), $a = 0_A$, so for every $s_i \in \{1_{A'}\}^{\infty\text{-sat}}$ (in particular, there is some such s_i) one has $a \cdot s_i = 0_A$.

Finally, given any $a' \in A'$, since φ is surjective, there is some element $a \in A$ such that $\varphi(a) = a'$. Since $1_A \in \{1_{A'}\}^{\infty\text{-sat}}$ and $a = \frac{a}{1_A}$, we have:

$$a' = \varphi(a) \cdot \varphi(1_A)^{-1}.$$

Since $\varphi : A \rightarrow A'$ satisfies (i), (ii) and (iii), the one-element family $\{\varphi : A \rightarrow A'\}$ co-covers A , so $\{\varphi^{\text{op}} : A' \rightarrow A\} \in \text{Cov}(A)$.

Stability axiom:

Now we are going to show that our definition of Cov is stable under pullbacks, that is:

If $(a_1, \dots, a_n) \in A \times A \times \dots \times A$ is a n -tuple such that $\langle a_1, \dots, a_n \rangle = A$ and $\{\eta_{a_i}^{\text{op}} : A\{a_i^{-1}\} \rightarrow A\}_{i=1, \dots, n}$ generates a covering family for A , then given a \mathcal{C}^∞ -rings homomorphism $g : A \rightarrow B$, since $\eta_{g(a_i)} \circ g$ is such that $(\eta_{g(a_i)} \circ g)(a_i) \in B\{g(a_i)^{-1}\}^\times$, by the universal property of $\eta_{a_i} : A \rightarrow A\{a_i^{-1}\}$ there is a unique \mathcal{C}^∞ -homomorphism:

$$g' : A\{a_i^{-1}\} \rightarrow B\{g(a_i)^{-1}\}$$

such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\eta_{a_i}} & A\{a_i^{-1}\} \\ g \downarrow & & \downarrow g' \\ B & \xrightarrow{\eta_{g(a_i)}} & B\{g(a_i)^{-1}\} \end{array}$$

By **Proposition 3.4**, the diagram above is a pushout, so

$$\begin{array}{ccc}
 B\{g(a_i)^{-1}\} & \xrightarrow{(\eta_{g(a_i)})^{\text{op}}} & B \\
 g'^{\text{op}} \downarrow & & \downarrow g^{\text{op}} \\
 A\{a_i^{-1}\} & \xrightarrow{(\eta_{a_i})^{\text{op}}} & A
 \end{array}$$

is a pullback in $\mathcal{C}^\infty \mathbf{Rng}_{\text{fp}}^{\text{op}}$.

In order to show that the family $\{(\eta_{g(a_i)})^{\text{op}} : B\{g(a_i)^{-1}\} \rightarrow B \mid i = 1, \dots, n\}$ belongs to $\text{Cov}(B)$, it suffices to show that $\{\eta_{g(a_i)} : B \rightarrow B\{g(a_i)^{-1}\} \mid i = 1, \dots, n\}$ belongs to $\text{coCov}(B)$.

Since a_1, \dots, a_n are such that $\langle a_1, \dots, a_n \rangle = A$, there are some $\lambda_1, \dots, \lambda_n \in A$ such that:

$$1_A = \sum_{i=1}^n \lambda_i \cdot a_i$$

Since $g : A \rightarrow B$ is a \mathcal{C}^∞ -homomorphism, we have:

$$1_B = g(1_A) = \sum_{i=1}^n g(\lambda_i) \cdot g(a_i),$$

thus $\langle g(a_1), \dots, g(a_n) \rangle = B$. Also, since for every $i = 1, \dots, n$, $\eta_{g(a_i)} : B \rightarrow B\{g(a_i)^{-1}\}$ is a \mathcal{C}^∞ -ring of fractions, it follows that $\{\eta_{g(a_i)} : B \rightarrow B\{g(a_i)^{-1}\} \mid i = 1, \dots, n\} \in \text{coCov}(B)$, hence:

$$\{(\eta_{g(a_i)})^{\text{op}} : B\{g(a_i)^{-1}\} \rightarrow B \mid i = 1, \dots, n\} \in \text{Cov}(B).$$

Transitivity axiom:

If $\{\eta_{a_i}^A : A \rightarrow A\{a_i^{-1}\} \mid i = 1, \dots, n\}$ generates a co-covering family of A and for each i , $\{\eta_{\beta_{ij}^{A_i}}^{A_i} : A\{a_i^{-1}\} \rightarrow (A\{a_i^{-1}\})\{\beta_{ij}^{-1}\} \mid j \in \{1, \dots, n_i\}\}$ generates a co-covering family of $A\{a_i^{-1}\}$, then:

$$\{\eta_{\beta_{ij}^{A_i}}^{A_{a_i}} \circ \eta_{a_i}^A : A \rightarrow (A\{a_i^{-1}\})\{\beta_{ij}^{-1}\} \mid i \in \{1, \dots, n\} \& j \in \{1, \dots, n_i\}\}$$

generates a co-covering family of A .

To show that the transitive axiom holds we will need the following technical result on ‘‘Smooth Commutative Algebra’’:

If for each $i \leq n$ and each $\beta_{ij} \in A\{a_i^{-1}\}$, $j \leq n_i$, we write $\beta_{ij} = \eta_a(b_{ij})/\eta_a(c_{ij})$, with $c_{ij} \in \{a_i\}^{\infty\text{-sat}}$, then by **Proposition 3.3**, to show that:

$$\{\eta_{\beta_{ij}}^{A_{a_i}} \circ \eta_{a_i}^A : A \rightarrow (A\{a_i^{-1}\})\{\beta_{ij}^{-1}\} | i \in \{1, \dots, n\} \& j \in \{1, \dots, n_i\}\}$$

generates a co-covering family of A amounts to show that:

$$\{\eta_{a_i \cdot b_{ij}}^A : A \rightarrow A\{(a_i \cdot b_{ij})^{-1}\} | (i \in \{1, \dots, n\}) \& (j \in \{1, \dots, n_i\})\}$$

does.

By hypothesis, $\{\eta_{a_i}^{\text{op}} : A\{a_i^{-1}\} \rightarrow A | i \in \{1, \dots, n\}\}$ generates a covering family of A , so:

$$1_A \in \langle a_1, \dots, a_n \rangle$$

or, equivalently:

$$\text{Spec}^\infty(A) = \bigcup_{i=1}^n D_A^\infty(a_i)$$

Since for every $i \in \{1, \dots, n\}$ we have a canonical homeomorphism:

$$\begin{aligned} \varphi : \text{Spec}^\infty(A\{a_i^{-1}\}) &\rightarrow D^\infty(a_i) \\ \mathfrak{p} &\mapsto \eta_{a_i}^{-1}[\mathfrak{p}] \end{aligned}$$

Also by hypothesis, for any $i \in \{1, \dots, n\}$, $\{(\eta_{a_i b_{ij}}^A)^{\text{op}} : A\{a_i \cdot b_{ij}^{-1}\} \rightarrow A\{a_i^{-1}\} | j \in \{1, \dots, n_i\}\}$ generates a covering family of $A\{a_i^{-1}\}$, so:

$$\text{Spec}^\infty(A\{a_i^{-1}\}) = \bigcup_{j=1}^{n_i} D_{A\{a_i^{-1}\}}^\infty(\beta_{ij}) \approx \bigcup_{j=1}^{n_i} D_A^\infty(a_i) \cap D_A^\infty(b_{ij}) \approx \bigcup_{j=1}^{n_i} D_A^\infty(a_i \cdot b_{ij})$$

Putting all together we obtain:

$$\text{Spec}^\infty(A) = \bigcup_{i=1}^n D_A^\infty(a_i) \approx \bigcup_{i=1}^n \text{Spec}^\infty(A\{a_i^{-1}\}) \approx \bigcup_{i=1}^n \left(\bigcup_{j=1}^{n_i} D_A^\infty(a_i \cdot b_{ij}) \right)$$

thus,

$$\text{Spec}^\infty(A) = \bigcup_{\substack{i \leq n \\ j \leq n_i}} D_A^\infty(a_i \cdot b_{ij})$$

but this is equivalent to

$$1_A \in \langle \{a_i \cdot b_{ij} : i \leq n, j \leq n_i\} \rangle$$

and the transitivity is proved.

Thus, Cov defines a Grothendieck pretopology on $\mathcal{C}^\infty \mathbf{Rng}_{\text{fp}}^{\text{op}}$. We have:

$$J_{\text{Cov}} : \text{Obj}(\mathcal{C}^\infty \mathbf{Rng}_{\text{fp}}) \rightarrow \wp(\wp(\text{Mor}(\mathcal{C}^\infty \mathbf{Rng}_{\text{fp}})))$$

given by:

$$J_{\text{Cov}}(A) := \{ \overleftarrow{S} \subseteq \cup_{B \in \text{Obj}(\mathcal{C}^\infty \mathbf{Rng}_{\text{fp}})} \text{Hom}_{\mathcal{C}^\infty \mathbf{Rng}_{\text{fp}}}(B, A) \mid S \in \text{Cov}(A) \}$$

turning $(\mathcal{C}^\infty \mathbf{Rng}_{\text{fp}}^{\text{op}}, J_{\text{Cov}})$ into a small site - the so called smooth Zariski site. ■

Proposition 3.11 *Let I be a finite set (say $I = \{1, \dots, n\}$) and let $\{A\{a_i^{-1}\} \xrightarrow{\eta_{a_i}} A \mid i \in I\}$ be a J_{Cov} -covering of A in $\mathcal{C}^\infty \mathbf{Rng}_{\text{fp}}^{\text{op}}$, then the diagram below is an equalizer in the category of \mathcal{C}^∞ -rings:*

(E)

$$A \rightarrow \prod_{i \in I} A\{a_i^{-1}\} \rightrightarrows \prod_{i, j \in I} A\{(a_i \cdot a_j)^{-1}\}$$

Proof. By hypothesis, $A = \langle \{a_i \mid i \in I\} \rangle$, or equivalently, $\text{Spec}^\infty(A) = D^\infty(1) = \bigcup_{i \in I} D^\infty(a_i)$.

Since the affine \mathcal{C}^∞ -locally ringed space of A , Σ_A , is in particular a sheaf of \mathcal{C}^∞ -rings, then the diagram below is an equalizer in the category of \mathcal{C}^∞ -rings.

$$\Sigma_A(D^\infty(1)) \longrightarrow \prod_{i \in I} \Sigma_A(D^\infty(a_i)) \rightrightarrows \prod_{i, j \in I} \Sigma_A(D^\infty(a_i) \cap D^\infty(a_j))$$

As $D^\infty(a_i) \cap D^\infty(a_j) = D^\infty(a_i \cdot a_j)$ and $\Sigma_A(D^\infty(b)) \cong A\{b^{-1}\}$, we have that the diagram of \mathcal{C}^∞ -rings below is an equalizer

$$A \xrightarrow{\cong} A\{1^{-1}\} \rightarrow \prod_{i \in I} A\{a_i^{-1}\} \rightrightarrows \prod_{i, j \in I} A\{(a_i \cdot a_j)^{-1}\}$$

and this finishes the proof. ■

We define the **smooth Grothendieck-Zariski topos**, that we denote by \mathcal{Z}^∞ , as the topos of sheaves over the smooth Zariski site:

$$\mathcal{Z}^\infty = \text{Sh}(\mathcal{C}^{\text{op}}, J_{\text{Cov}}),$$

where \mathcal{C} is a skeleton of the category of all finitely presented \mathcal{C}^∞ -rings, $\mathcal{C}^\infty \mathbf{Rng}_{\text{f.p.}}$.

Remark 3.12 *The forgetful functor $\mathcal{O} : \mathcal{C}^\infty\mathbf{Rng}_{\text{fp}} \rightarrow \mathbf{Sets}$ is called the structure sheaf of the Grothendieck-Zariski smooth topos.*

This is actually a sheaf of sets since if $\{\eta_{a_i} : A \rightarrow A\{a_i^{-1}\} \mid i \leq n\}$ is a smooth Zariski co-covering (i.e. $A = \langle \{a_1, \dots, a_n\} \rangle$), then the diagram of sets below must be an equalizer,

$$A \rightarrow \prod_{i \in I} A\{a_i^{-1}\} \rightrightarrows \prod_{i, j \in I} A\{(a_i \cdot a_j)^{-1}\}$$

since it is indeed an equalizer of \mathcal{C}^∞ -rings and the forgetful functor $\mathcal{C}^\infty\mathbf{Rng} \rightarrow \mathbf{Sets}$ preserves limits.

Proposition 3.13 *The following rectangle is a pushout:*

$$\begin{array}{ccc} A & \xrightarrow{\eta_{a_i}} & A\{a_i^{-1}\} \\ \eta_{a_j} \downarrow & \dashrightarrow^{\eta_{a_i \cdot a_j}} & \downarrow \\ A\{a_j^{-1}\} & \longrightarrow & A\{(a_i \cdot a_j)^{-1}\} \end{array}$$

Theorem 3.14 *The smooth Grothendieck-Zariski topology J_{Cov} on \mathcal{Z}^∞ is sub-canonical, that is, for every finitely presented \mathcal{C}^∞ -ring B , the representable functor:*

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(\bullet, B) : & \mathcal{C}^{\text{op}} & \rightarrow & \mathbf{Set} \\ & A & \mapsto & \text{Hom}_{\mathcal{C}}(A, B) \\ & (A_1 \xrightarrow{f} A_2) & \mapsto & (\text{Hom}_{\mathcal{C}}(A_2, B) \xrightarrow{-\circ f} \text{Hom}_{\mathcal{C}}(A_1, B)) \end{array}$$

is a sheaf (of sets).

Proof. Let I be a finite set (lets say $I = \{1, \dots, n\}$) and let $\{A\{a_i^{-1}\} \xrightarrow{\eta_{a_i}} A \mid i \in I\}$ be a \mathcal{Z}^∞ -covering of A in \mathcal{C} .

Recall, from **Proposition 3.13**, that for every $i, j \in I$, the following rectangle is a pushout in $\mathcal{C}^\infty\mathbf{Rng}$:

$$\begin{array}{ccc} A & \xrightarrow{\eta_{a_i}} & A\{a_i^{-1}\} \\ \eta_{a_j} \downarrow & \dashrightarrow^{\eta_{a_i \cdot a_j}} & \downarrow \\ A\{a_j^{-1}\} & \longrightarrow & A\{(a_i \cdot a_j)^{-1}\} \end{array}$$

We must prove that:

(I)

$$\mathcal{C}(A, B) \rightarrow \prod_{i \in I} \mathcal{C}(A\{a_i^{-1}\}, B) \rightrightarrows \prod_{i, j \in I} \mathcal{C}(A\{(a_i \cdot a_j)^{-1}\}, B)$$

is an equalizer diagram of sets and functions.

Since $\mathcal{C} = \mathcal{C}^\infty \mathbf{Rng}_{\text{fp}}^{\text{op}}$, this amounts to prove that

(II)

$$\mathcal{C}^\infty \mathbf{Rng}_{\text{fp}}(B, A) \rightarrow \prod_{i \in I} \mathcal{C}^\infty \mathbf{Rng}_{\text{fp}}(B, A\{a_i^{-1}\}) \rightrightarrows \prod_{i, j \in I} \mathcal{C}^\infty \mathbf{Rng}_{\text{fp}}(B, A\{(a_i \cdot a_j)^{-1}\})$$

is an equalizer diagram of sets and functions.

As Hom functors preserve products, the diagram (II) is isomorphic to

(III)

$$\mathcal{C}^\infty \mathbf{Rng}_{\text{fp}}(B, A) \rightarrow \mathcal{C}^\infty \mathbf{Rng}_{\text{fp}}(B, \prod_{i \in I} A\{a_i^{-1}\}) \rightrightarrows \mathcal{C}^\infty \mathbf{Rng}_{\text{fp}}(B, \prod_{i, j \in I} A\{(a_i \cdot a_j)^{-1}\})$$

But this is an equalizer diagram of sets and functions since the Hom functor $\mathcal{C}^\infty \mathbf{Rng}_{\text{fp}}(B, -)$ preserves equalizers and the diagram

(E)

$$A \rightarrow \prod_{i \in I} A\{a_i^{-1}\} \rightrightarrows \prod_{i, j \in I} A\{(a_i \cdot a_j)^{-1}\}$$

is an equalizer in the category of \mathcal{C}^∞ -rings, by **Proposition 3.11**.

Thus the Grothendieck topology J_{Cov} of the smooth Zariski site is sub-canonical. ■

Now we show that the topos of sheaves on the smooth Zariski site, that we have just described, is the classifying topos of the theory of the \mathcal{C}^∞ -local rings.

In order to define a “local \mathcal{C}^∞ -ring object” in a topos, we use - as motivation - the Mitchell-Bénabou language. We define a local \mathcal{C}^∞ -ring object in a topos \mathcal{E} as follows: it is a \mathcal{C}^∞ -ring object R in \mathcal{E} such that the (geometric) formula:

$$(\forall a \in R)((\exists b \in R)(a \cdot b = 1) \vee (\exists b \in R)((1 - a) \cdot b = 1))$$

is valid.

By definition, this means that the union of the subobjects:

$$\begin{aligned} \{a \in R \mid \exists b \in R(a \cdot b = 1)\} &\twoheadrightarrow R, \\ \{a \in R \mid \exists b \in R((1 - a) \cdot b = 1)\} &\twoheadrightarrow R \end{aligned}$$

of R is all of R . Equivalently, consider the two subobjects of the product $R \times R$ defined by:

$$\begin{cases} U = \{(a, b) \in R \times R \mid a \cdot b = 1\} \twoheadrightarrow R \times R \\ V = \{(a, b) \in R \times R \mid (1 - a) \cdot b = 1\} \twoheadrightarrow R \times R \end{cases} \tag{9}$$

The \mathcal{C}^∞ -ring object R is local if, and only if, the two composites $U \twoheadrightarrow R \times R \xrightarrow{\pi_1} R$ and $V \twoheadrightarrow R \times R \xrightarrow{\pi_1} R$ form an epimorphic family in \mathcal{E} .

In **Section 2**, we have observed that there is an equivalence between \mathcal{C}^∞ -ring objects R in a topos \mathcal{E} and left exact functors, $\mathcal{C}^\infty\mathbf{Rng}_{\text{fp}}^{\text{op}} \rightarrow \mathcal{E}$. Explicitly, given such a left-exact functor F , the corresponding \mathcal{C}^∞ -ring object R in \mathcal{E} is $F(\mathcal{C}^\infty(\mathbb{R}))$. Conversely, given a \mathcal{C}^∞ -ring R in \mathcal{E} , the corresponding functor:

$$\phi_R : \mathcal{C}^\infty\mathbf{Rng}_{\text{fp}}^{\text{op}} \rightarrow \mathcal{E}$$

sends the finitely presented \mathcal{C}^∞ -ring $A = \frac{\mathcal{C}^\infty(\mathbb{R}^n)}{\langle p_1, \dots, p_k \rangle}$ to the following equalizer in \mathcal{E} :

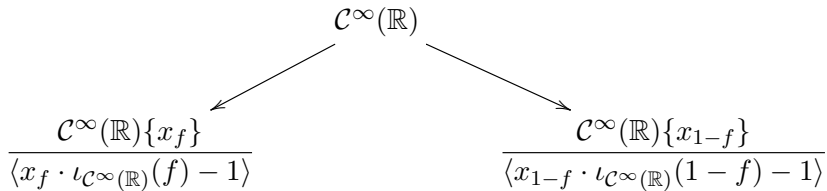
$$\phi_R(A) \twoheadrightarrow R^n \begin{array}{c} \xrightarrow{(p_1, \dots, p_k)} \\ \xrightarrow{(0, \dots, 0)} \end{array} R^k \tag{10}$$

This description readily yields the corresponding definition of ϕ_R on arrows.

The following lemma gives a condition for a \mathcal{C}^∞ -ring R in a topos \mathcal{E} to be local, phrased in terms of this corresponding functor ϕ_R .

Lemma 3.15 *Let \mathcal{E} be a topos, R be a \mathcal{C}^∞ -ring object in \mathcal{E} , and let $\phi_R : \mathcal{C}^\infty\mathbf{Rng}_{\text{fp}}^{\text{op}} \rightarrow \mathcal{E}$ be the corresponding left exact functor. The following are equivalent:*

- (i) R is a local \mathcal{C}^∞ -ring in \mathcal{E} ;
- (ii) ϕ_R sends the pair of arrows in the category $\mathcal{C}^\infty\mathbf{Rng}_{\text{fp}}$:



to an epimorphic family of two arrows in \mathcal{E} ;

(iii) For any finitely presented \mathcal{C}^∞ -ring A and any elements a_1, \dots, a_n such that $\langle a_1, \dots, a_n \rangle = A$, ϕ_R sends the family of arrows in $\mathcal{C}^\infty\mathbf{Rng}_{\text{fp}}$:

$$\{A \rightarrow A\{a_i^{-1}\} \mid i = 1, \dots, n\}$$

to an epimorphic family $\{\phi_R(A\{a_i^{-1}\}) \rightarrow \phi_R(A) \mid i = 1, \dots, n\}$ in \mathcal{E} .

Proof. Ad (i) \iff (ii): This follows immediately from the explicit description of the functor ϕ_R . Let

$$A = \frac{\mathcal{C}^\infty(\mathbb{R})\{x_f\}}{\langle x_f \cdot \iota_{\mathcal{C}^\infty(\mathbb{R})}(f) - 1 \rangle}$$

and

$$B = \frac{\mathcal{C}^\infty(\mathbb{R})\{x_{1-f}\}}{\langle x_{1-f} \cdot \iota_{\mathcal{C}^\infty(\mathbb{R})}(1-f) - 1 \rangle}$$

Note that:

$$\phi_R(A) = \{(a, b) \in R \times R \mid a \cdot b = 1\}$$

To wit, ϕ_R sends $A = \frac{\mathcal{C}^\infty(\mathbb{R})\{x_f\}}{\langle x_f \cdot \iota_{\mathcal{C}^\infty(\mathbb{R})}(f) - 1 \rangle}$ to the equalizer:

$$\phi_R(A) \xrightarrow{\quad} R \times R \xrightleftharpoons[1 \circ!]{x_f \cdot \iota_{\mathcal{C}^\infty(\mathbb{R})}(f)} R$$

and ϕ_R sends B to:

$$\phi_R(B) \xrightarrow{\quad} R \times R \xrightleftharpoons[1 \circ!]{x_{1-f} \cdot \iota_{\mathcal{C}^\infty(\mathbb{R})}(1-f)} R$$

The arrow $\mathcal{C}^\infty(\mathbb{R}) \rightarrow A$ is mapped into the composite $\alpha : \phi_R(A) \rightarrow R$ given by $\phi_R(A) \rightarrow R \times R \xrightarrow{\pi_1} R$, and the arrow $\mathcal{C}^\infty(\mathbb{R}) \rightarrow B$, is mapped into the composite $\beta : \phi_R(B) \rightarrow R$, given by $\phi_R(B) \rightarrow R \times R \xrightarrow{\pi_1} R$.

By the definition of a local \mathcal{C}^∞ -ring, (i) is equivalent to (ii).

Ad (ii) \implies (2): is also clear, since (ii) is the special case of (iii) in which $A = \mathcal{C}^\infty(\mathbb{R})$ while $n = 2$, $a_1 = f$ and $a_2 = 1 - f$.

Ad (ii) \implies (iii): Assume that (ii) holds, and suppose given a finitely presented \mathcal{C}^∞ -ring A and elements $a_1, \dots, a_n \in A$ with $\sum_{i=1}^n a_i = 1$. This result is proved using induction. We are going to prove the:

Case $n = 2$.

In this case $a_2 = 1 - a_1$. We form the pushouts of $\mathcal{C}^\infty(\mathbb{R}) \rightarrow \frac{\mathcal{C}^\infty(\mathbb{R})\{x_f\}}{\langle x_f \cdot \iota_{\mathcal{C}^\infty(\mathbb{R})}(f) - 1 \rangle}$ and $\mathcal{C}^\infty(\mathbb{R}) \rightarrow \frac{\mathcal{C}^\infty(\mathbb{R})\{x_{1-f}\}}{\langle x_{1-f} \cdot \iota_{\mathcal{C}^\infty(\mathbb{R})}(1-f) - 1 \rangle}$ along the map $\mathcal{C}^\infty(\mathbb{R}) \rightarrow A$ sending $\text{id}_{\mathbb{R}}$ to a_1 , as in:

$$\begin{array}{ccccc} \frac{\mathcal{C}^\infty(\mathbb{R})\{x_{1-f}\}}{\langle x_{1-f} \cdot \iota_{\mathcal{C}^\infty(\mathbb{R})}(1-f) - 1 \rangle} & \longleftarrow & \mathcal{C}^\infty(\mathbb{R}) & \longrightarrow & \frac{\mathcal{C}^\infty(\mathbb{R})\{x_f\}}{\langle x_f \cdot \iota_{\mathcal{C}^\infty(\mathbb{R})}(f) - 1 \rangle} \\ \downarrow & & \downarrow a_1 & & \downarrow \\ A\{(1 - a_1)^{-1}\} & \longleftarrow & A & \longrightarrow & A\{a_1^{-1}\} \end{array}$$

giving the indicated \mathcal{C}^∞ -rings of fractions $A\{(1 - a_1)^{-1}\}$ or $A\{a_1^{-1}\}$. These squares are pullbacks in $\mathcal{C}^\infty\mathbf{Rng}_{\text{fp}}^{\text{op}}$, hence they are sent by the left-exact functor ϕ_R to pullbacks in \mathcal{E} , as in:

$$\begin{array}{ccccc} \phi_R \left(\frac{\mathcal{C}^\infty(\mathbb{R})\{x_{1-f}\}}{\langle x_{1-f} \cdot \iota_{\mathcal{C}^\infty(\mathbb{R})}(1-f) - 1 \rangle} \right) & \xrightarrow{\pi_1} & R & \xleftarrow{\pi_1} & \phi_R \left(\frac{\mathcal{C}^\infty(\mathbb{R})\{x_f\}}{\langle x_f \cdot \iota_{\mathcal{C}^\infty(\mathbb{R})}(f) - 1 \rangle} \right) \\ \uparrow & & \uparrow & & \uparrow \\ \phi_R(A\{(1 - a_1)^{-1}\}) & \longrightarrow & \phi_R(A) & \longleftarrow & \phi_R(A\{a_1^{-1}\}) \end{array}$$

But by assumption

$$\phi_R \left(\frac{\mathcal{C}^\infty(\mathbb{R})\{x_{1-f}\}}{\langle x_{1-f} \cdot \iota_{\mathcal{C}^\infty(\mathbb{R})}(1-f) - 1 \rangle} \right) \rightarrow R$$

and

$$\phi_R \left(\frac{\mathcal{C}^\infty(\mathbb{R})\{x_f\}}{\langle x_f \cdot \iota_{\mathcal{C}^\infty(\mathbb{R})}(f) - 1 \rangle} \right) \rightarrow R$$

form an epimorphic family in \mathcal{E} , and hence so does the pullback of this family. This proves (iii) for the case $n = 2$.

The general case follows by induction. For instance, if $n = 3$ and $a_1 + a_2 + a_3 = 1$, let $\beta \in A\{(a_2 + a_3)^{-1}\}$ such that $\beta \cdot \eta(a_2) + \beta \cdot \eta(a_3) = 1$. Then, again by the case $n = 2$, ϕ_R sends the three arrows in $\mathcal{C}^\infty\mathbf{Rng}_{\text{fp}}$

$$\begin{aligned} A &\rightarrow A\{a_1^{-1}\} \\ A &\rightarrow A\{a_2^{-1}\} \rightarrow A\{(a_2 + a_3)^{-1}\}\{(\beta \cdot \eta(a_2))^{-1}\} \\ A &\rightarrow A\{a_3^{-1}\} \rightarrow A\{(a_2 + a_3)^{-1}\}\{(\beta \cdot \eta(a_3))^{-1}\} \end{aligned}$$

to an epimorphic family in \mathcal{E} . Thus ϕ_R also sends the family of canonical arrows $\{A \rightarrow A\{a_i^{-1}\} : i = 1, 2, 3\}$ to an epimorphic family in \mathcal{E} . ■

Theorem 3.16 *The smooth Grothendieck-Zariski topos $\mathcal{Z}^\infty = \text{Sh}(\mathcal{C}^{\text{op}}, J_{\text{Cov}})$, is a classifying topos for local \mathcal{C}^∞ -rings, i.e., for any co-complete topos \mathcal{E} , there is an equivalence of categories:*

$$\text{Geom}(\mathcal{E}, \mathcal{Z}^\infty) \simeq \mathcal{C}^\infty\text{LocRng}(\mathcal{E}) \tag{11}$$

where $\mathcal{C}^\infty\text{LocRng}(\mathcal{E})$ is the category of local \mathcal{C}^∞ -ring-objects in \mathcal{E} .

The universal local \mathcal{C}^∞ -ring is the structure sheaf \mathcal{O} of the Grothendieck-Zariski smooth topos (see **Remark 3.12**).

Proof. As a special case of the results on classifying topoi presented in the Section 1, there is an equivalence between $\text{Geom}(\mathcal{E}, \mathcal{Z}^\infty)$ and the category of continuous left-exact functors from $\mathcal{C}^\infty\mathbf{Rng}_{\text{fp}}$ to \mathcal{E} .

This category is equivalent to the full subcategory $\mathcal{C}^\infty\mathbf{LocRng}(\mathcal{E})$ consisting of local \mathcal{C}^∞ -rings.

The identification of the universal local \mathcal{C}^∞ -ring is the object of \mathcal{Z}^∞ represented by the object $\mathcal{C}^\infty(\mathbb{R})$ of the Grothendieck Zariski smooth site, this is precisely the structure sheaf (= forgetful functor) $\mathcal{O} : \mathcal{C}^\infty\mathbf{Rng}_{\text{fp}} \rightarrow \mathbf{Sets}$. ■

4 A Classifying Topos for the Theory of the von Neumann-regular \mathcal{C}^∞ -rings

A von Neumann regular \mathcal{C}^∞ -ring is a \mathcal{C}^∞ -ring A such that one of the (following) equivalent conditions hold:

- (i) $(\forall a \in A)(\exists x \in A)(a = a^2x)$;
- (ii) Every principal ideal of A is generated by an idempotent element, i.e.,

$$(\forall a \in A)(\exists e \in A)(\exists y \in A)(\exists z \in A)((e^2 = e) \& (ey = a) \& (az = e))$$

- (iii) $(\forall a \in A)(\exists! b \in A)((a = a^2b) \& (b = b^2a))$

For a proof of this result in the setting of usual commutative rings, see, for instance, [1].

Whenever A is a \mathcal{C}^∞ -reduced (i.e., $\sqrt[\infty]{(0_A)} = (0_A)$) von Neumann regular \mathcal{C}^∞ -ring, $\text{Spec}^\infty(A)$ is a Boolean space. In an upcoming paper ([5]) we show that part of the converse is true: for a fixed \mathcal{C}^∞ -field, given any Boolean space

(X, τ) , there is some \mathcal{C}^∞ -reduced von Neumann regular \mathcal{C}^∞ -ring, R_X such that $\text{Spec}^\infty(R_X) \approx X$.

Now we turn to the problem of constructing a classifying topos for this \mathcal{C}^∞ -rings. As defined above, a von Neumann-regular \mathcal{C}^∞ -ring is a \mathcal{C}^∞ -rings (A, Φ) in which the first-order formula:

$$(\forall x \in A)(\exists! y \in A)((xyx = x) \& (yxy = y)) \tag{12}$$

holds. Denoting by $\varphi(x, y) := ((xyx = x) \& (yxy = y))$, we note that the formula:

$$(\forall x \in A)(\exists! y \in A)\varphi(x, y)$$

defines a functional relation from A to A , so we can define an unary functional symbol.

Let \mathbb{T}_{vN} be the theory of the von Neumann-regular \mathcal{C}^∞ -rings in the language \mathbb{L} described at the beginning of the first section of the first chapter. We can define the unary functional symbol $*$ by means of the formula (12):

$$\begin{aligned} * : A &\rightarrow A \\ x &\mapsto y \text{ s.t. } \varphi(x, y) \end{aligned}$$

in order to obtain a richer language, namely $\mathbb{L}' = \mathbb{L} \cup \{*\}$.

Remark 4.1 (a) Note that in every von Neumann-regular \mathcal{C}^∞ -ring V , since $x^*xx^* = x^*$ holds for every $x \in V$, then $0^* = 0$.

(b) If \mathbb{F} is a \mathcal{C}^∞ -field, thus a von Neumann-regular \mathcal{C}^∞ -ring, so

$$\mathbb{F} \models \sigma.$$

Since $xx^*x = x$ holds for every $x \in \mathbb{F}$, then if $x \neq 0$, we must have

$$x^* = \frac{1}{x}.$$

(c) In fact, the unary function $*$ does not belong to the language \mathbb{L} .

We have seen that $\mathcal{C}^\infty(\mathbb{R}^0) \cong \mathbb{R}$, together with its canonical \mathcal{C}^∞ -structure Φ , is a \mathcal{C}^∞ -field, thus a von Neumann-regular \mathcal{C}^∞ -ring, so

$$\mathbb{R} \models \sigma.$$

Now, the function:

$$\begin{aligned} * : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto \begin{cases} \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

is defined by σ . However, there is no (continuous, and thus) smooth function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$(\forall x \in \mathbb{R})(x^* = f(x)),$$

that is,

$$(\forall f \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}))(\Phi(f) \neq *)$$

so $*$ is not a symbol of the original language of \mathcal{C}^∞ -rings.

Remark 4.2 Let \mathbb{T}' be the a theory in the language $\mathbb{L}' = \mathbb{L} \cup \{*\}$, that contains:

- the (equational) \mathbb{L} -axioms for of \mathcal{C}^∞ -rings;
- the (equational) \mathbb{L}' -axiom

$$\sigma := (\forall x)((xx^*x = x) \& (x^*xx^* = x^*))$$

that is, $\mathbb{T}' := \mathbb{T} \cup \{\sigma\}$. By the **Theorem of Extension by Definition** (cf. **Corollary 4.4.7** of [9]), we know that \mathbb{T}' is conservative extension of \mathbb{T} .

Remark 4.3 (a) Since $\varphi(x, y)$ is a conjunction of two equations, the von Neumann-regular \mathcal{C}^∞ -homomorphisms preserve $*$, i.e.,

$$(\forall x \in \mathbb{R})(h(x^*) = h(x)^*)$$

whenever (A, Φ) and (B, Ψ) are von Neumann-regular \mathcal{C}^∞ -rings and $h : (A, \Phi) \rightarrow (B, \Psi)$ is a von Neumann-regular \mathcal{C}^∞ -homomorphism.

(b) Since the \mathbb{L} -class of von Neumann-regular \mathcal{C}^∞ -rings is closed under quotients by \mathcal{C}^∞ -congruences and \mathcal{C}^∞ -congruences are classified by ideals, it follows from the item (a) that for each von Neumann-regular \mathcal{C}^∞ -ring V and any ideal $I \subseteq V$, then $x^* - y^* \in I$ whenever $x - y \in I$.

Definition 4.4 A finitely presented von Neumann regular \mathcal{C}^∞ -ring is a von Neumann-regular \mathcal{C}^∞ -ring (V, Φ) such that there is a finite set X and an ideal $I \subseteq L(X) = \mathbf{vN}(\mathcal{C}^\infty(\mathbb{R}^X))$ with:

$$V \cong \frac{\mathbf{vN}(\mathcal{C}^\infty(\mathbb{R}^X))}{I}$$

Remark 4.5 (V, Φ) is a finitely presented von Neumann-regular \mathcal{C}^∞ -ring if, and only if the representable functor:

$$\mathbf{Hom}_{\mathcal{C}^\infty \mathbf{vNRng}}(V, \bullet) : \mathcal{C}^\infty \mathbf{vNRng} \rightarrow \mathbf{Set}$$

preserves all directed colimits. That is to say that for every directed system of von Neumann-regular \mathcal{C}^∞ -rings $\{(V_i, \Phi_i), \nu_{ij} : (V_i, \Phi_i) \rightarrow (V_j, \Phi_j)\}_{i,j \in I}$ we have

$$\text{Hom}_{\mathcal{C}^\infty\mathbf{vNRng}}(V, \varinjlim_{i \in I} V_i) = \varinjlim_{i \in I} \text{Hom}_{\mathcal{C}^\infty\mathbf{vNRng}}(V, V_i),$$

(cf. **Proposition 3.8.14** of [6])

Consider the category whose objects are all finitely presented von Neumann regular \mathcal{C}^∞ -ring and whose morphisms are the \mathcal{C}^∞ -homomorphisms between them, and denote it by $\mathcal{C}^\infty\mathbf{vNRng}_{\text{f.p.}}$.

Remark 4.6 We have:

$$\text{Obj}(\mathcal{C}^\infty\mathbf{Rng}_{\text{fp}}) \cap \text{Obj}(\mathcal{C}^\infty\mathbf{vNRng}) \subseteq \mathcal{C}^\infty\mathbf{vNRng}_{\text{fp}}.$$

Thus, keeping in mind the remarks above, and following the same line of the developments made in the section 2 we obtain:

Theorem 4.7 The category

$$\mathcal{C}^\infty\mathbf{vNRng}_{\text{fp}}^{\text{op}}$$

is a category with finite limits freely generated by the von Neumann regular \mathcal{C}^∞ -ring $\mathbf{vN}(\mathcal{C}^\infty(\mathbb{R}))$, i.e., for any category with finite limits \mathcal{C} , the evaluation of a left-exact functor $F : \mathcal{C}^\infty\mathbf{vNRng}_{\text{fp}}^{\text{op}} \rightarrow \mathcal{C}$ at $\mathbf{vN}(\mathcal{C}^\infty(\mathbb{R}))$ yields the following equivalence of categories:

$$\begin{aligned} \text{ev}_{\mathbf{vN}(\mathcal{C}^\infty(\mathbb{R}))} : \text{Lex}(\mathcal{C}^\infty\mathbf{vNRng}_{\text{fp}}^{\text{op}}, \mathcal{C}) &\rightarrow \mathcal{C}^\infty\mathbf{vNRng}(\mathcal{C}) \\ F &\mapsto F(\mathbf{vN}(\mathcal{C}^\infty(\mathbb{R}))) \end{aligned}$$

Combining the results presented in this section and the one stated in the **section 1** on classifying topoi, we obtain the following:

Theorem 4.8 The presheaf topos $\mathbf{Sets}^{\mathcal{C}^\infty\mathbf{vNRng}_{\text{fp}}}$ is a classifying topos for von Neumann regular \mathcal{C}^∞ -rings, and the universal von Neumann regular \mathcal{C}^∞ -ring R is the von Neumann regular \mathcal{C}^∞ -ring object in $\mathbf{Sets}^{\mathcal{C}^\infty\mathbf{vNRng}_{\text{fp}}}$ given by the inclusion functor from $\mathcal{C}^\infty\mathbf{vNRng}_{\text{fp}}$ to $\mathcal{C}^\infty\mathbf{vNRng}$. Thus, for any Grothendieck topos \mathcal{E} there is an equivalence of categories, natural in \mathcal{E} :

$$\begin{aligned} \text{Geom}(\mathcal{E}, \mathbf{Sets}^{\mathcal{C}^\infty\mathbf{vNRng}_{\text{fp}}}) &\rightarrow \mathcal{C}^\infty\mathbf{vNRng}(\mathcal{E}) \\ f &\mapsto f^*(R) \end{aligned}$$

5 Final remarks and future works

We have described classifying toposes for three theories: the theory of \mathcal{C}^∞ -rings and the theories of local and of von Neumann regular \mathcal{C}^∞ -rings. In [15], I. Moerdijk, N. van Quê and G. Reyes present the classifying topos for the (geometric) theory of Archimedean \mathcal{C}^∞ -rings. This reinforces the following questions:

- Are there other sensible descriptions of classifying toposes for other distinguished classes of \mathcal{C}^∞ -rings?
- In particular, is there a nice description of the theory of von Neumann regular \mathcal{C}^∞ -rings in the language of \mathcal{C}^∞ -rings (without the need for the new symbol for the “quasi-inverse”)?

In the paper (under preparation) [5], we use von Neumann regular \mathcal{C}^∞ -rings in order to classify Boolean algebras. We show that a von Neumann regular \mathcal{C}^∞ -ring is isomorphic to the \mathcal{C}^∞ -ring of global sections of the structure sheaf of its affine \mathcal{C}^∞ -scheme. Such results motivate us to look for similar characterizations for some distinguished classes of \mathcal{C}^∞ -rings in terms of its \mathcal{C}^∞ -spectrum topology.

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