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On Extensions of Isomorphisms of Substructures

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Abstract

Let $\mathcal{L}$ be a first order relational language with identity and let $\mathcal{L}_{\alpha\beta}$ be the usual infinitary extension of $\mathcal{L}$. Given an $\mathcal{L}$-structure $\mathcal{E}$ and two substructures $\mathcal{F}_1, \mathcal{F}_2$ of $\mathcal{E}$; an $\mathcal{L}_{\alpha\beta}$-strong isomorphism of $\mathcal{F}_1$ and $\mathcal{F}_2$ is an isomorphism $f : \mathcal{F}_1 \to \mathcal{F}_2$ which preserves the intersections of the $\mathcal{L}_{\alpha\beta}$-definable relations of $\mathcal{E}$ (Definition 4.1). For a suitable choice of α, β; a necessary and sufficient condition for f to be extendable to an automorphism of $\mathcal{E}$ is that f be $\mathcal{L}_{\alpha\beta}$-strong (Theorem 4.2). If every isomorphism between substructures of $\mathcal{E}$ is $\mathcal{L}_{\alpha\beta}$-strong for an adequate choice of α and β, it follows that $\mathcal{E}$ is homogeneous (Theorem 4.5). The result is used to prove that, for any $\mathcal{L}$-structure $\mathcal{E}$, quantifier elimination in a suitable language $\mathcal{L}_{\alpha\beta}$ implies homogeneity, whatever the cardinality of $\mathcal{E}$ (Corollary 5.3).

Keywords: infinitary languages, strong isomorphisms, homogeneous structures.

1 Infinitary languages

Let α, β be infinite cardinal numbers, α being a regular cardinal and $\beta \leq \alpha$. By $\mathcal{L}_{\alpha\beta}$ we denote an *infinitary first order relational language*, whose formulas are sequences of less than α symbols and which admits sequences of less than α conjunctions and blocks of sequences of less than β instantiations.

In order to fix our notation, we recall the rules of formation of formulas. We denote by $|A|$ the cardinal of a set A and by $\wp(A)$ its power set and use freely the notion of concatenation of sequences. (See [2].)

The symbols of $\mathcal{L}_{\alpha\beta}$ are the logical symbols $\neg, \bigwedge, \exists$, relation symbols including $=$ and symbols of variables. We use the abbreviations $\forall, \bigvee$ and the standard convention for the use of parenthesis. We assume that the cardinal of the set V of variables is α and that we have a fixed enumeration $\chi : i \in \alpha \to x_i \in V$ of V and consider V as an ordered set: $x_i \leq x_j$ if and only if $i \leq j$. The set of relation symbols is denoted by $\mathcal{R}$; to each relation symbol $R \in \mathcal{R}$ is associated a natural number $ar(R)$ called the *arity* of R.

The set of *formulas* of $\mathcal{L}_{\alpha\beta}$ is the smallest set of sequences of symbols of length less than α satisfying the following conditions:

1.1) If R is a relation symbol of arity n and $\tau : n \to V$ is a sequence of variables, then $R\tau$ is a formula;

1.2) If φ is a formula, $\neg\varphi$ is a formula;

1.3) If $\gamma < \alpha$ is an ordinal number and $(\varphi_i)_{i<\gamma}$ is a sequence of formulas, then $\bigwedge(\varphi_i)_{i<\gamma}$ is a formula;

1.4) If φ is a formula, $\gamma < \beta$ is an ordinal number and $\eta : \gamma \to V$ is a sequence of variables, then $\exists\eta\varphi$ is a formula.

We also use the notation $\mathcal{L}_{\alpha\beta}$ to denote the set of formulas of the language $\mathcal{L}_{\alpha\beta}$. The set $V(\varphi)$ of free variables of a formula φ is defined as usual. The *arity* of φ is by definition the ordinal number of $V(\varphi)$ endowed with the order induced by the order of V. If φ is a formula of arity γ, we denote by $\sigma_\varphi : \gamma \to V(\varphi)$ the order preserving bijection. The language $\mathcal{L}_{\omega\omega}$ is the standard finitary first order relational language and will be denoted simply by $\mathcal{L}$.

2 Relational structures

Given a set E and an ordinal γ, a γ-tuple of points of E is a sequence $p : \gamma \to E$ of elements of E defined on γ. We refer to γ-tuples also as γ-points or points of arity γ defined on E. Denote by E^γ the set of all γ-points defined on E; a relation of arity γ is a subset of E^γ.

Given an $\mathcal{L}_{\alpha\beta}$-language and a non empty set E, an $\mathcal{L}_{\alpha\beta}$-*relational structure* $\mathcal{E}$ defined on E is a pair $(E, \mathcal{R}_\mathcal{E})$ such that $\mathcal{R}_\mathcal{E}$ is a map that assigns to each predicate symbol $R \in \mathcal{R}$ of arity n a relation $R_\mathcal{E}$ of arity n defined on E; E is the *domain* of $\mathcal{E}$.

If a finitary relational language $\mathcal{L}$ has been fixed and given cardinals α, β, α regular and $\beta \leq \alpha$, we shall denote by $\mathcal{L}_{\alpha\beta}$ the infinitary language which has the same predicate symbols as $\mathcal{L}$ and whose ordered set of variables extend the set of variables of $\mathcal{L}$, that is , if x_i , $i < \alpha$, are the variables of $\mathcal{L}_{\alpha\beta}$, then x_i, $i < \omega$, are the variables of $\mathcal{L}$. We call $\mathcal{L}_{\alpha\beta}$ the extended language of $\mathcal{L}$. If $\mathcal{E}$ is an $\mathcal{L}$-structure, then $\mathcal{E}$ is also an $\mathcal{L}_{\alpha\beta}$-structure.

An *interpretation of the variables* of an $\mathcal{L}_{\alpha\beta}$-structure $\mathcal{E}$ is a map $\mathcal{I} : V \to E$. We recall the notion of an interpretation $\mathcal{I}$ of variables satisfying a formula φ in an $\mathcal{L}_{\alpha\beta}$-structure $\mathcal{E}$, denoted by $\mathcal{I} \vDash_\mathcal{E} \varphi$, using our notation:

2.1) If φ is $R\tau$, then $\mathcal{I} \vDash_\mathcal{E} \varphi \Leftrightarrow \mathcal{I} \circ \tau \in R_\mathcal{E}$;

2.2) If φ is $\neg\psi$, then $\mathcal{I} \vDash_\mathcal{E} \varphi \Leftrightarrow \mathcal{I} \nvDash_\mathcal{E} \psi$ ($\mathcal{I}$ does not satisfy ψ in $\mathcal{E}$);

2.3) If φ is $\bigwedge(\varphi_i)_{i<\gamma}$, then $\mathcal{I} \vDash_\mathcal{E} \varphi \Leftrightarrow \mathcal{I} \vDash_\mathcal{E} \varphi_i$ for all $i < \gamma$;

2.4) If φ is $\exists\eta\psi$, then $\mathcal{I} \vDash_{\mathcal{E}} \varphi$ if and only if there exists $\mathcal{I}' : V \to E$ such that $\mathcal{I}' \vDash_{\mathcal{E}} \psi$ and $\mathcal{I}(x_i) = \mathcal{I}'(x_i)$ for every $x_i \in V(\varphi)$, that is, $\mathcal{I} \circ \sigma_\varphi = \mathcal{I}' \circ \sigma_\varphi$.

Let $\varphi \in \mathcal{L}_{\alpha\beta}$ be a formula of arity γ of an $\mathcal{L}_{\alpha\beta}$-structure $\mathcal{E}$. The relation $[\varphi]_{\mathcal{E}}$ defined by φ in the structure $\mathcal{E}$ is the set of all points $p \in E^\gamma$ for which there exists an interpretation $\mathcal{I}$ of the variables satisfying φ and $p = \mathcal{I} \circ \sigma_\varphi$, that is:

$$[\varphi]_{\mathcal{E}} = \{\mathcal{I} \circ \sigma_\varphi \in E^\gamma : \mathcal{I} \vDash_{\mathcal{E}} \varphi\}.$$

A relation R defined on E is *definable* in the $\mathcal{L}_{\alpha\beta}$-structure $\mathcal{E}$ if there exists a formula $\varphi \in \mathcal{L}_{\alpha\beta}$ such that $R = [\varphi]_{\mathcal{E}}$.

3 Invariance and definability

Let $\mathcal{E}$ be an $\mathcal{L}$-structure over the domain E and denote by G the group of automorphisms of $\mathcal{E}$ and by $\bar{\delta}$ the cardinal of the set $\wp(\delta)$ where $\delta = |E|$.

The action of G on E extends to an action of G on E^γ, where γ is any ordinal. If $g \in G$ and $p \in E^\gamma$, $g \cdot p = g \circ p$. For any choice of α, β, any relation definable in the extended $\mathcal{L}_{\alpha\beta}$-structure $\mathcal{E}$ is invariant under G.

Theorem 3.1 *Let $p \in E^\delta$ be a bijection of δ onto E. The orbit $G \cdot p$ of p under the action of G is the intersection S of all $\mathcal{L}_{\bar{\delta}\bar{\delta}}$-definable relations that contains p.*

Proof. Since S is invariant by G and $p \in S$, it follows that $G \cdot p \subset S$.

Let $N \subset E^\delta$ be the set of all bijections of δ onto E. The following formula of $\mathcal{L}_{\bar{\delta}\bar{\delta}}$, where y is a variable distinct from all $x_i, i < \delta$, defines N,

$$\forall y[\bigvee_{i<\delta}(y = x_i)] \wedge [\bigwedge_{i,j<\delta, i\neq j}(x_i \neq x_j)].$$

Hence $S \subset N$.

Let q be any point of S and consider the bijection $g = q \cdot p^{-1} : E \to E$. We have $g \cdot p = q$; we shall prove that $g \in G$.

Since $\bar{\delta}$ is a regular cardinal, $|\mathcal{L}_{\bar{\delta}\bar{\delta}}| < \bar{\delta}$. Hence, the power of the set of definable relations of arity $\leq \delta$ in the $\mathcal{L}_{\bar{\delta}\bar{\delta}}$-structure $\mathcal{E}$ is less then $\bar{\delta}$, yielding that S is definable in the $\mathcal{L}_{\bar{\delta}\bar{\delta}}$-structure. Let $\varphi \in \mathcal{L}_{\bar{\delta}\bar{\delta}}$ be a formula that defines S and assume that the free variables of φ are $(x_i)_{i<\delta}$. Let R be a relation symbol of $\mathcal{L}$ of arity n and denote also by $R \subset E^n$ the relation represented by the symbol R. Given a point $a \in R$, we define the map $\eta = (\chi|\delta) \circ p^{-1} \circ a : n \to V$ and consider the following formula of $\mathcal{L}_{\bar{\delta}\bar{\delta}}$:

$$\psi = \varphi \wedge R\eta.$$

Let $\mathcal{I} : V \to E$ be an interpretation of variables satisfying, for $i < \delta$, $\mathcal{I}(x_i) = p_i$. Since $\sigma_\varphi = \chi|\delta$, we have $\mathcal{I} \circ \sigma_\varphi = p$ and from $p \in [\varphi]_\mathcal{E}$ it follows that $\mathcal{I} \models_\mathcal{E} \varphi$. From $\mathcal{I} \circ \eta = p \circ (\chi|\delta)^{-1} \circ (\chi|\delta) \circ p^{-1} \circ a \in R$ it follows that $\mathcal{I} \models_\mathcal{E} R\eta$; hence, $\mathcal{I} \models_\mathcal{E} \psi$. From $\sigma_\psi = \chi|\delta$ and $p = \mathcal{I} \circ \sigma_\psi$ we have that $p \in [\psi]_\mathcal{E}$. Thus, there exists $\mathcal{I}' : V \to E$ such that $q = \mathcal{I}' \circ \chi|\delta$ and $\mathcal{I}' \circ \eta = \mathcal{I}' \circ (\chi|\delta) \circ p^{-1} \circ a \in R$; hence, $q \circ p^{-1} \circ a \in R$ or, equivalently, $g \cdot a \in R$. This proves that $g \cdot R \subset R$; interchanging the rules of p and q we have for $g^{-1} = p^{-1} \circ q$, $g^{-1} \cdot R \subset R$. Therefore, $g \cdot R = R$. Since R is any primitive relation of $\mathcal{E}$, this proves that $g \in G$. Since q is any point of S, we have $G \cdot p \supset S$, completing the proof that $G \cdot p = S$. ■

Corollary 3.2 *Let $p : \gamma \to E, \gamma \leq \delta$, be an injection. The orbit $G \cdot p \subset E^\gamma$ of p is $\mathcal{L}_{\bar{\delta}\bar{\delta}}$-definable.*

Proof. If $\gamma = \delta$ and $p : \delta \to E$ is a bijection, by theorem 3.1 the orbit $G \cdot p$ is $\mathcal{L}_{\bar{\delta}\bar{\delta}}$-definable. If $\gamma < \delta$, the injection p can be extended to a bijection $q : \delta \to E$. Let $\psi \in \mathcal{L}_{\bar{\delta}\bar{\delta}}$ be a formula which defines $G \cdot q$ and assume that the free variables of ψ are $(x_i)_{i<\delta}$. Then, the formula $\exists (x_i)_{\gamma \leq i < \delta}\, \psi$ defines the orbit $G \cdot p$. ■

Remark 3.3 *Based on corollary 3.2, it is not difficult to prove that the orbit of any point $p \in E^\gamma, \gamma \leq \delta$ is $\mathcal{L}_{\bar{\delta}\bar{\delta}}$-definable. Since any relation of E^γ, invariant under the action of G is a union of orbits of G, it follows that every invariant relation of $\mathcal{E}$ of arity $\gamma \leq \delta$ is $\mathcal{L}_{\bar{\delta}\bar{\delta}}$-definable. See [4] and [1].*

4 Strong isomorphisms and homogeneity

Let $\mathcal{E}_i$ be $\mathcal{L}$-structures defined on domains E_i; and let $\mathcal{F}_i$ be substructures of $\mathcal{E}_i$ defined on domains $F_i \subset E_i$, such that $|F_i| < |E_i|$, $i = 1, 2$. We consider $\mathcal{E}_i$ also as $\mathcal{L}_{\alpha\beta}$-structures where $\mathcal{L}_{\alpha\beta}$ is the extended language of $\mathcal{L}$.

If γ is an ordinal number, for any map $f : E_1 \to E_2$, we denote by $f^\gamma : E_1^\gamma \to E_2^\gamma$ the natural extension of f to E_1^γ; if $p \in E_1^\gamma$, $f^\gamma(p) = f \circ p$.

Definition 4.1 *An isomorphism $f : F_1 \to F_2$ of the $\mathcal{L}$-structures $\mathcal{F}_1$ and $\mathcal{F}_2$ is $\mathcal{L}_{\alpha\beta}$-*strong *if for all formulas $\varphi \in \mathcal{L}_{\alpha\beta}$ of arity γ we have*

$$f^\gamma([\varphi]_{\mathcal{E}_1} \cap F_1^\gamma) = [\varphi]_{\mathcal{E}_2} \cap F_2^\gamma \qquad (\Xi)$$

If f is the restriction to F_1 of an isomorphism $\tilde{f} : E_1 \to E_2$ of the $\mathcal{L}$-structures $\mathcal{E}_1$ and $\mathcal{E}_2$, then f is $\mathcal{L}_{\alpha\beta}$-strong for any choice of α, β.

Theorem 4.2 *Let $\mathcal{E}_i$ be isomorphic $\mathcal{L}$-structures with domains E_i, and let $\mathcal{F}_i$ be a substructure of $\mathcal{E}_i$, with domains F_i, $i = 1, 2$. Denote by $\bar{\delta}$ the cardinal of the set $\wp(\delta)$ where $\delta = |E_1| = |E_2|$. If $|F_1| < \delta$, then any $\mathcal{L}_{\bar{\delta}\bar{\delta}}$-strong isomorphism of $\mathcal{F}_1$ and $\mathcal{F}_2$ can be extended to an isomorphism of $\mathcal{E}_1$ and $\mathcal{E}_2$.*

Proof. Let $f : F_1 \to F_2$ be an $\mathcal{L}_{\bar{\delta}\bar{\delta}}$-strong isomorphism of $\mathcal{F}_1$ and $\mathcal{F}_2$. Consider an isomorphism $h : E_1 \to E_2$ of $\mathcal{E}_1$ and $\mathcal{E}_2$ and a bijection $p : \gamma \to F_1$, where $\gamma = |F_1|$. We extended p to a bijection $q : \delta \to E_1$. By theorem 3.1, there are formulas $\varphi, \tilde{\varphi} \in \mathcal{L}_{\bar{\delta}\bar{\delta}}$ which define the orbits $G \cdot p, G \cdot q$ of p and q under the action of the group G of automorphisms of $\mathcal{E}_1$. We assume that $V(\varphi) = \{x_i : i < \gamma\}$ and $V(\tilde{\varphi}) = \{x_i : i < \delta\}$. Consider the following formula of $\mathcal{L}_{\bar{\delta}\bar{\delta}}$:

$$\psi = \exists (x_i)_{i \in \delta - \gamma} (\varphi \wedge \tilde{\varphi}).$$

Clearly $p \in [\psi]_{\mathcal{E}_1}$. By definition of strong isomorphism, we have:

$$f^\gamma([\psi]_{\mathcal{E}_1} \cap F_1^\gamma) = [\psi]_{\mathcal{E}_2} \cap F_2^\gamma.$$

It follows that $f^\gamma(p) \in [\psi]_{\mathcal{E}_2}$. Hence, for each i, $\gamma \leq i < \delta$, there exists $c(i) \in E_2$ verifying the condition that the sequence $r_2 : \delta \to E_2$ defined by $r_2(i) = f(q(i))$ if $i < \gamma$ and $r_2(i) = c(i)$ if $\gamma \leq i < \delta$ belongs to $[\tilde{\varphi}]_{\mathcal{E}_2}$. Since h^δ maps $[\tilde{\varphi}]_{\mathcal{E}_1}$ onto $[\tilde{\varphi}]_{\mathcal{E}_2}$, there exists $r_1 \in [\tilde{\varphi}]_{\mathcal{E}_1}$ such that $h^\delta(r_1) = r_2$. By definition of $\tilde{\varphi}$, there exists $g \in G$ satisfying $g \cdot q = r_1$. Consider the isomorphism $\tilde{f} = h \circ g : E_1 \to E_2$ of $\mathcal{E}_1$ and $\mathcal{E}_2$. For every element $a \in F_1$, there exists $i < \gamma$ verifying $p(i) = q(i) = a$. Then, $\tilde{f}(a) = h(g(a)) = h(g(q(i))) = h(r_1(i)) = r_2(i) = f(q(i)) = f(a)$. Therefore, $\tilde{f}$ extends f. ■

Definition 4.3 *Let $\mathcal{E}$ be an $\mathcal{L}$-structure defined on the domain E, assume $|E| = \delta$. We say that $\mathcal{E}$ has the* property of strong isomorphisms *if every isomorphism between substructures of $\mathcal{E}$ is $\mathcal{L}_{\bar{\delta}\bar{\delta}}$-strong.*

Definition 4.4 *Let $\mathcal{E}$ be an $\mathcal{L}$-structure defined on the domain E and assume that $|E| = \delta$. $\mathcal{E}$ is* homogeneous *if any isomorphism of substructures of $\mathcal{E}$, of cardinal less than δ, can be extended to an automorphism of $\mathcal{E}$.*

Theorem 4.5 *An $\mathcal{L}$-structure $\mathcal{E}$ has the property of strong isomorphism if and only if $\mathcal{E}$ is homogeneous.*

Proof. Immediate from theorem 4.2. ■

Theorem 4.5 states a different form to characterize homogeneous $\mathcal{L}$-structures considering infinitary entensions of $\mathcal{L}$.

5 Quantifier elimination and categoricity

Let $\mathcal{E}$ be an $\mathcal{L}$-structure and consider two formulas φ and ψ of the extended language $\mathcal{L}_{\alpha\beta}$. We say that φ and ψ are *equivalent in the $\mathcal{L}_{\alpha\beta}$-structure $\mathcal{E}$* (or *$\mathcal{E}$-equivalent*) if $[\varphi]_{\mathcal{E}} = [\psi]_{\mathcal{E}}$.

We say that $\mathcal{E}$ has *quantifier elimination* in $\mathcal{L}_{\alpha\beta}$ if every formula of $\mathcal{L}_{\alpha\beta}$ which has free variables is $\mathcal{E}$-equivalent to an $\mathcal{L}_{\alpha\beta}$-formula without quantifiers.

Theorem 5.1 *Let $\mathcal{E}_i$ be isomorphic $\mathcal{L}$-structures, $i = 1, 2$. Given cardinals α, β, assume that $\mathcal{E}_1$ has quantifier elimination in $\mathcal{L}_{\alpha\beta}$. Then, every isomorphism $f : F_1 \to F_2$ of substructures $\mathcal{F}_i$ of $\mathcal{E}_i$, $i = 1, 2$, is $\mathcal{L}_{\alpha\beta}$-strong.*

Proof. By definition of substructures, equality (Ξ), see definition 4.1, holds for every atomic formula φ of $\mathcal{L}$. It is easily proved that if (Ξ) holds for a formula φ, then it holds also for $\neg\varphi$. Let φ_i, $i < \gamma < \alpha$ be a sequence of quantifiers-free formulas for which (Ξ) holds; we shall prove below that (Ξ) also holds for $\bigwedge(\varphi_i)_{i<\gamma}$ and $\bigvee(\varphi_i)_{i<\gamma}$.

To prove that (Ξ) holds for $\varphi = \bigwedge(\varphi_i)_{i<\gamma}$, denote by V_i the set of free variables of φ_i and by $V = \bigcup_{i<\gamma} V_i$ the set of free variables of φ. For each $i < \gamma$, consider the formula

$$\tilde{\varphi}_i = \varphi_i \wedge \bigwedge_{x_k \in V - V_i} x_k = x_k,$$

and assume that the arity of φ is γ_0. It is easily verified that (Ξ) holds for $\tilde{\varphi}_i$. Then, we have:

$$\begin{aligned}
h^{\gamma}([\varphi]_{\mathcal{E}_1} \cap F_1^{\gamma_0}) &= h^{\gamma_0}([\textstyle\bigwedge_{i<\gamma} \varphi_i]_{\mathcal{E}_1} \cap F_1^{\gamma_0}) \\
&= h^{\gamma_0}(\textstyle\bigcap_{i<\gamma}([\tilde{\varphi}_i]_{\mathcal{E}_1}) \cap F_1^{\gamma_0}) \\
&= h^{\gamma_0}(\textstyle\bigcap_{i<\gamma}([\tilde{\varphi}_i]_{\mathcal{E}_1} \cap F_1^{\gamma_0})) \\
&= \textstyle\bigcap_{i<\gamma} h^{\gamma_0}([\tilde{\varphi}_i]_{\mathcal{E}_1} \cap F_1^{\gamma_0}) \\
&= \textstyle\bigcap_{i<\gamma}([\tilde{\varphi}_i]_{\mathcal{E}_2} \cap F_2^{\gamma_0}) \\
&= (\textstyle\bigcap_{i<\gamma}([\tilde{\varphi}_i]_{\mathcal{E}_2})) \cap F_2^{\gamma_0} \\
&= [\textstyle\bigwedge_{i<\gamma} \varphi_i]_{\mathcal{E}_2} \cap F_2^{\gamma_0} \\
&= [\varphi]_{\mathcal{E}_2} \cap F_2^{\gamma_0}.
\end{aligned}$$

The proof for $\bigvee(\varphi_i)_{i<\gamma}$ is similar and will be ommited. ■

The next theorem is a model-theoretical generalization of well known theorems of theories of algebraically closed fields and differentially closed fields of characteristic zero. For instance, the Steinitz's isomorphism theorem: *Let K and K' be fields, and let L, L' be, respectively, algebraic closures of K and K'. Then, every isomorphism from K to K' is extendible to an isomorphism from L to L'.*

Theorem 5.2 *Let $\mathcal{E}_i$ be isomorphic $\mathcal{L}$-structures over domains E_i, $i = 1,2$; that have quantifier elimination in $\mathcal{L}_{\bar{\delta}\bar{\delta}}$, $\bar{\delta} = |\wp(E_1)|$. Assume that $\mathcal{F}_i$ is a substructure of $\mathcal{E}_i$ whose domain F_i satisfies $|F_i| < |E_i|$, $i = 1, 2$. Any isomorphism of substructures $\mathcal{F}_1$ and $\mathcal{F}_2$ admits an extension to an isomorphism of $\mathcal{E}_1$ and $\mathcal{E}_2$.*

Proof. Theorem 5.2 is a straightforward consequence of the theorems 4.2 and 5.1. ■

Corollary 5.3 *An $\mathcal{L}$-structure $\mathcal{E}$ is homogeneous if it has quantifier elimination in $\mathcal{L}_{\bar{\delta}\bar{\delta}}$, $\bar{\delta} = |\wp(E)|$.*

We recall that an $\mathcal{L}$-structure $\mathcal{E}$ is *categorical* if any $\mathcal{L}$-structure $\mathcal{E}'$, $|E| = |E'|$, which is elementary equivalent to $\mathcal{E}$, is isomorphic to $\mathcal{E}$.

The following theorem is easily proved using compacity. If the cardinality of E is ω, the proof can be found in [3], corollary 3.1.3 and proposition 3.1.6.

Theorem 5.4 *Let $\mathcal{E}$ be a homogeneous $\mathcal{L}$-structure. If either $|\mathcal{R}|$ is finite or $|\mathcal{R}| \leq |E|$ and $\mathcal{E}$ is categorical, then $\mathcal{E}$ has quantifier elimination.*

The theorem bellow is a consequence of theorems 4.5 and 5.4.

Theorem 5.5 *Let $\mathcal{E}$ be an $\mathcal{L}$-structure. If either $|\mathcal{R}|$ is finite or $|\mathcal{R}| \leq |E|$ and $\mathcal{E}$ is categorical, then the following statements are equivalent:*

1. *$\mathcal{E}$ has quantifier elimination in $\mathcal{L}_{\bar{\delta}\bar{\delta}}$, $\bar{\delta} = |\wp(E)|$;*
2. *$\mathcal{E}$ has the property of strong isomorphisms;*
3. *$\mathcal{E}$ is homogeneous.*

6 Final remarks

As far as the authors know, the notion of strong isomorphisms, stated in a set theoretical way, is due to J. S. e Silva. See [5].

Although Silva did not have the notion of infinitary languages, his result in [5], pp 112-113 is, in essence, equivalent to theorem 4.2. See also [1], theorem 7.1 p 23.

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