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Remarks on Propositional Logics and the Categorial Relationship Between Institutions and Π-Institutions

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Abstract

In this work we explore some applications of the notions of *Institution* and π-*Institution* in the setting of *propositional logics* and establish a precise categorial relation between these notions. Specifically, we provide a pair of functors that establish an adjunction between the categories **Inst** and π-**Inst**.

1 Introduction

The notion of *Institution* was introduced for the first time by Goguen and Burstall in [**GB**]. This concept formalizes the notion of logical system into a mathematical object, i.e., it provides a *"...categorical abstract model theory which formalizes the intuitive notion of logical system, including syntax, semantic, and satisfaction relation between them..."* [**Diac**]. This means that it encompasses the abstract concept of universal model theory for a logic. The main (model-theoretical) characteristic is that an institution contains a satisfaction relation between models and sentences that are coherent under change of notation. First-order (infinitary) logics with Tarski's semantics are natural examples of institutions (see section 2.1).

A variation of the formalism of institutions, the notion of π-*Institution*, were defined by Fiadeiro and Sernadas in [**FS**] providing an alternative (proof-theoretical) approach to deductive system, which *"...replace the notion of model and satisfaction by a primitive consequence operator (à la Tarski)"* [**FS**]. Natural categories of propositional logics (see section 2.2) provide examples of π-institutions.

In [**FS**] and [**Vou**] was established a relation between institutions and π-institutions. To the best of our knowledge, there is no literature on categorial connections between the category of institutions and the category of

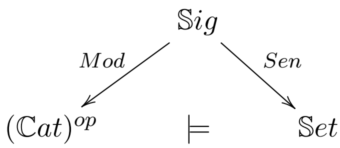
π-institutions. In the section 2.3 of the present work, we provide a precise categorial relationship between these notions, that extends the above mentioned relation between objects of those categories, more precisely, we determine a pair of adjoint functors between those categories. We finish this work with some remarks concerning, mainly, applications of these tools to the propositional logic setting.

2 The categories Inst and π − Inst

We start giving the definition of institution and π-institution with their respective notions of morphisms (and comorphisms), and consequently their categories.

2.1 Institution and its category

Definition 2.1 *An Institution $I = (\mathbb{S}ig, Sen, Mod, \models)$ consists of*



1. *a category $\mathbb{S}ig$, whose objects are called* signatures*,*
2. *a functor $Sen : \mathbb{S}ig \to \mathbb{S}et$, for each signature a set whose elements are called* sentences *over the signature*
3. *a functor $Mod : (\mathbb{S}ig)^{op} \to \mathbb{C}at$, for each signature a category whose objects are called* models*,*
4. *a relation $\models_\Sigma \subseteq |Mod(\Sigma)| \times Sen(\Sigma)$ for each $\Sigma \in |\mathbb{S}ig|$, called Σ-*satisfaction*, such that for each morphism $h : \Sigma \to \Sigma'$, the compatibility condition*

$$M' \models_{\Sigma'} Sen(h)(\phi) \quad \textit{if and only if} \quad Mod(h)(M') \models_\Sigma \phi$$

holds for each $M' \in |Mod(\Sigma')|$ and $\phi \in Sen(\Sigma)$

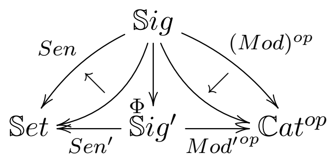
Example 2.2 *Let Lang denote the category of languages $L = ((F_n)_{n\in\mathbb{N}}, (R_n)_{n\in\mathbb{N}})$, – where F_n is a set of symbols of n-ary function symbols and R_n is a set of symbols of n-ary relation symbols, $n \geq 0$ – and language morphisms*[1]*. For each*

[1] That can be chosen "strict" (i.e., $F_n \mapsto F'_n$, $R_n \mapsto R'_n$) or chosen be "flexible" (i.e., $F_n \mapsto \{n - ary - terms(L')\}$, $R_n \mapsto \{n - ary - atomic - formulas(L')\}$).

pair of cardinals $\aleph_0 \leq \kappa, \lambda \leq \infty$, *the category Lang endowed with the usual notion of* $L_{\kappa,\lambda}$*-sentences (=* $L_{\kappa,\lambda}$*-formulas with no free variable), with the usual association of category of structures and with the usual (tarskian) notion of satisfaction, gives rise to an institution* $I(\kappa, \lambda)$.

Definition 2.3 *Let* I *and* I' *be institutions.*

(a) An Institution **morphism** $h = (\Phi, \alpha, \beta) : I \to I'$ *consists of*



1. *a functor* $\Phi : \mathbb{S}ig \to \mathbb{S}ig'$

2. *a natural transformation* $\alpha : Sen' \circ \Phi \Rightarrow Sen$

3. *a natural transformation* $\beta : Mod \Rightarrow Mod' \circ \Phi^{op}$

such that the following compatibility condition *holds:*

$$m \models_\Sigma \alpha_\Sigma(\varphi') \quad \textit{iff} \quad \beta_\Sigma(m) \models'_{\Phi(\Sigma)} \varphi'$$

for any $\Sigma \in \mathbb{S}ig$, *any* Σ*-model* m *and any* $\Phi(\Sigma)$*-sentence* φ'.

(b) A triple $f = \langle \phi, \alpha, \beta \rangle : I \to I'$ *is a* **comorphism** *between the given institutions if the following conditions hold:*

- $\phi : \mathbb{S}ig \to \mathbb{S}ig'$ *is a functor.*
- $\alpha : Sen \Rightarrow Sen' \circ \phi$ *and* $\beta : Mod' \circ \phi^{op} \Rightarrow Mod$ *are natural transformations such that satisfy:*

$$m' \models'_{\phi(\Sigma)} \alpha_\Sigma(\varphi) \quad \textit{iff} \quad \beta_\Sigma(m') \models_\Sigma \varphi$$

for any $\Sigma \in \mathbb{S}ig$, $m' \in Mod'(\phi(\Sigma))$ *and* $\varphi \in Sen(\Sigma)$.

Example 2.4 *Given two pairs of cardinals* (κ_i, λ_i), *with* $\aleph_0 \leq \kappa_i, \lambda_i \leq \infty$, $i = 0, 1$, *such that* $\kappa_0 \leq \kappa_1$ *and* $\lambda_0 \leq \lambda_1$, *then it is induced a morphism and a comorphism of institutions* $(\Phi, \alpha, \beta) : I(\kappa_0, \lambda_0) \to I(\kappa_1, \lambda_1)$, *given by the* same data*:* $\mathbb{S}ig_0 = Lang = \mathbb{S}ig_1$, $Mod_0 = Mod_1 : (Lang)^{op} \to \mathbb{C}at$, $Sen_i = L_{\kappa_i, \lambda_i}$, $i = 0, 1$, $\Phi = Id_{Lang} : \mathbb{S}ig_0 \to \mathbb{S}ig_1$, $\beta := Id : Mod_i \Rightarrow Mod_{1-i}$, $\alpha := inclusion : Sen_0 \Rightarrow Sen_1$.

Given $f : I \to I'$ and $f' : I \to I''$ comorphisms of institutions, then $f' \bullet f := \langle \phi' \circ \phi, \alpha' \bullet \alpha, \beta' \bullet \beta \rangle$ defines a comorphism $f' \bullet f : I \to I''$, where $(\alpha' \bullet \alpha)_\Sigma = \alpha'_{\phi(\Sigma)} \circ \alpha_\Sigma$ and $(\beta' \bullet \beta)_\Sigma = \beta_\Sigma \circ \beta'_{\phi(\Sigma)}$. Let $Id_I := \langle Id_{\mathbb{S}ig}, Id, Id \rangle : I \to I$. It is straightforward to check that these data determines a category[2]. We will denote by **Inst** this category of institutions where the arrows are **comorphisms** of institutions. Of course, it can also be formed a category whose objects are institutions and the arrows are **morphisms** of institutions, but that will be less important here.

2.2 π-Institutions and its category

Definition 2.5 *A π-Institution $J = \langle \mathbb{S}ig, Sen, \{C_\Sigma\}_{\Sigma \in |\mathbb{S}ig|} \rangle$ is a triple with its first two components exactly the same as the first two components of an institution and, for every $\Sigma \in |\mathbb{S}ig|$, a closure operator $C_\Sigma : \mathcal{P}(Sen(\Sigma)) \to \mathcal{P}(Sen(\Sigma))$, such that the following coherence conditions holds, for every $f : \Sigma_1 \to \Sigma_2 \in Mor(\mathbb{S}ig)$:*

$$Sen(f)(C_{\Sigma_1}(\Gamma)) \subseteq C_{\Sigma_2}(Sen(f)(\Gamma)), \ \textit{for all } \Gamma \subseteq Sen(\Sigma_1).$$

Definition 2.6 *Given π-institutions J and J', $g = \langle \phi, \alpha \rangle : J \to J'$ is a* ***comorphism*** *between π-institutions when the following conditions hold:*

- *$\phi : \mathbb{S}ig \to \mathbb{S}ig'$ is a functor*
- *$\alpha : Sen \Rightarrow Sen' \circ \phi$ is a natural transformation such that satisfies the compatibility condition:*

$$\varphi \in C_\Sigma(\Gamma) \Rightarrow \alpha_\Sigma(\varphi) \in C_{\phi(\Sigma)}(\alpha_\Sigma(\Gamma)) \ \ \textit{for all} \ \ \Gamma \cup \{\varphi\} \subseteq \mathbb{S}ig(\Sigma).$$

Let $g : J \to J'$ and $g' : J' \to J''$ be comorphisms of π-institutions. $g' \bullet g$ is defined as the two first components of composition of comorphisms of institutions. The identity (co)morphism is given as the two first components of the comorphism identity of institutions. We will denote by π-**Inst** the category of π-institutions and with comorphisms as its arrows.

Example 2.7 *In [**AFLM**], [**FC**], [**FC1**] and [**MaMe**] are considered some categories of propositional logics – $l = (\Sigma, \vdash)$, where $\Sigma = (\Sigma_n)_{n \in \mathbb{N}}$ is a finitary signature and $\vdash \subseteq P(Form(\Sigma)) \times Form(\Sigma))$ is a tarskian consequence*

[2] As usual in category theory, the set theoretical size issues on such global constructions of categories can be addressed by the use of, at least, two Grothendieck's universes.

operator– with morphisms $f : (\Sigma, \vdash) \to (\Sigma', \vdash')$, *some kind of signature morphism* $f : \Sigma \to \Sigma'$ *–"strict" or "flexible"– that induces a translation or interpretation :* $\Gamma \cup \{\psi\} \subseteq Form(\Sigma)$, $\Gamma \vdash \psi \Rightarrow \check{f}[\Gamma] \vdash' \check{f}(\psi)$.

*(a) To the category of propositional logics endowed with "*flexible *morphisms"* $\mathcal{L}_f$ *(respectively, endowed with "*strict *morphisms"* $\mathcal{L}_s$*) is associated a* π*-institution* J_f *(respectively,* J_s*) in the following way:*

- $\mathbb{S}ig_f := \mathcal{L}_f$;
- $Sen_f : \mathbb{S}ig_f \to \mathbb{S}et$, *given by* $(f : (\Sigma, \vdash) \to (\Sigma', \vdash)) \mapsto (\check{f} : F_\Sigma(X) \to F_{\Sigma'}(X))$;
- *For each* $l = (\Sigma, \vdash) \in |\mathbb{S}ig_f|$, $C_l : P(F_\Sigma(X)) \to P(F_\Sigma(X))$ *is given by* $C_l(\Gamma) := \{\phi \in F_\Sigma(X) \ : \ \Gamma \vdash_l \phi\}$, *for each* $\Gamma \subseteq F_\Sigma(X)$.

*(b) In [***MaMe***], the "inclusion" functor* $(+)_L : \mathcal{L}_s \to \mathcal{L}_f$ *induces a comorphism (and also a morphism!) of the associated* π*-institutions* $(+) := ((+)_L, \alpha^+) : J_s \to J_f$, *where, for each* $l = (\Sigma, \vdash) \in \mathbb{S}ig_s = \mathcal{L}_s$, $\alpha^+(l) = Id_{F_\Sigma(X)} : F_\Sigma(X) \to F_\Sigma(X)$.

2.3 An adjunction between Inst and π-Inst

In order to establish the adjunction between $Inst$ and $\pi - Inst$ we introduce the following:

Let $I = \langle \mathbb{S}ig, Sen, Mod, \models \rangle$ be an institution. Given $\Sigma \in |\mathbb{S}ig|$, consider

$$\Gamma^\star = \{m \in Mod(\Sigma) \ : \ m \models_\Sigma \varphi \ \textit{for all} \ \varphi \in \Gamma\} \ \ \textit{and}$$

$$M^\star = \{\varphi \in Sen(\Sigma) \ : \ m \models_\Sigma \varphi \ \textit{for all} \ m \in M\}$$

for any $\Gamma \subseteq Sen(\Sigma)$ and $M \subseteq Mod(\Sigma)$. Clearly, these mappings establish a Galois connection. Thus, $C^I_\Sigma(\Gamma) := \Gamma^{\star\star}$ defines a closure operator for any $\Sigma \in |\mathbb{S}ig|$ ([**Vou**]).

The following lemma describes the behavior of these Galois connections through institutions comorphisms.

Lemma 2.8 *Let* $f = \langle \phi, \alpha, \beta \rangle : I \to I'$ *be an arrow in* **Inst**. *Then, given* $\Gamma \subseteq Sen(\Sigma)$ *and* $M \subseteq |Mod(\phi(\Sigma))|$, *the following conditions holds:*

1) $\beta_\Sigma[(\alpha_\Sigma[\Gamma])^\star] \subseteq \Gamma^\star$;

2) $\alpha_\Sigma[(\beta_\Sigma[M])^\star] \subseteq M^\star$.

Proof. *1)* Let $m \in \beta_\Sigma[(\alpha_\Sigma[\Gamma])^\star]$. So, there is $m' \in \alpha_\Sigma[\Gamma]^\star$ such that $\beta_\Sigma(m') = m$. As $m' \in \alpha_\Sigma[\Gamma]^\star$, hence $m' \models'_{\phi(\Sigma)} \alpha_\Sigma[\Gamma] \ \Leftrightarrow \ \beta_\Sigma(m') \models_\Sigma \Gamma \ \Leftrightarrow \ m \models_\Sigma \Gamma$. Then $m \in \Gamma^\star$.

2) Let $\varphi \in \alpha_\Sigma[(\beta_\Sigma[M])^\star]$. So, there is $\psi \in \beta_\Sigma[M]^\star$such that $\alpha_\Sigma(\psi) = \varphi$. Since $\psi \in (\beta_\Sigma[M])^\star$, hence $\beta_\Sigma[m] \models_\Sigma \psi \ \Leftrightarrow \ m \models_{\phi(\Sigma)} \alpha_\Sigma(\psi) \ \Leftrightarrow \ m \models_{\phi(\Sigma)} \varphi$

for any $m \in M$. Therefore $\varphi \in M^\star$. ■

Define the following application:

$$\begin{array}{rccc} F: & \mathbf{Inst} & \longrightarrow & \pi - \mathbf{Inst} \\ & I & \longmapsto & F(I) = \langle \mathbb{S}ig, Sen, \{C^I_\Sigma\}_{\Sigma \in |\mathbb{S}ig|} \rangle \end{array}$$

In order to show that F is well-defined, it is enough to prove the compatibility condition for $\{C^I_\Sigma\}_{\Sigma\in|\mathbb{S}ig|}$, i.e., given $f : \Sigma_1 \to \Sigma_2$ and $\Gamma \subseteq Sen(\Sigma_1)$, then $Sen(f)(C^I_{\Sigma_1}(\Gamma)) \subseteq C^I_{\Sigma_2}(Sen(f)(\Gamma))$. Let $\varphi_2 \in Sen(f)(C^I_{\Sigma_1}(\Gamma))$, then there is $\varphi_1 \in \Gamma^{**}$ such that $Sen(f)(\varphi_1) = \varphi_2$. Let $m \in (Sen(f)(\Gamma))^*$. So $m \models_{\Sigma_2} Sen(f)(\Gamma)$. By compatibility condition in institutions we have that $Mod(f)(m) \models_{\Sigma_1} \Gamma$, thus $Mod(f)(m) \in \Gamma^*$. Since $\varphi_1 \in \Gamma^{**}$ we have that $Mod(f)(m) \models_{\Sigma_1} \varphi_1$, hence $m \models_{\Sigma_2} Sen(f)(\varphi_1) = \varphi_2$. Therefore

$$\varphi_2 \in (Sen(f)(\Gamma))^{**} = C^I_{\Sigma_2}(Sen(f)(\Gamma)).$$

Now, let $f = \langle \phi, \alpha, \beta \rangle : I \to I'$ be a comorphism of institutions. Then consider $F(f) = \langle \phi, \alpha \rangle$. Notice that $F(f)$ is a comorphism between $F(I)$ and $F(I')$. Indeed, it is enough to prove that $F(f)$ satisfies the compatibility condition. Let $\Gamma \cup \{\varphi\} \subseteq Sen(\Sigma)$ for some $\Sigma \in |\mathbb{S}ig|$. Suppose that $\alpha_\Sigma(\varphi) \notin C^I_{\phi(\Sigma)}(\alpha_\Sigma[\Gamma])$. Hence $\alpha_\Sigma(\varphi) \notin \alpha_\Sigma[\Gamma]^{\star\star}$. Therefore $\alpha_\Sigma[\Gamma]^\star \not\models'_{\phi(\Sigma)} \alpha_\Sigma(\alpha)$. Thus there is $m \in \alpha_\Sigma[\Gamma]^\star$ such that $m \not\models'_{\phi(\Sigma)} \alpha_\Sigma(\varphi)$. Hence $\beta_\Sigma(m) \not\models_\Sigma \varphi$. Due to Lemma 2.8 1) we have that $\beta_\Sigma(m) \in \Gamma^\star$. Therefore $\varphi \notin \Gamma^{\star\star} = C^I_\Sigma(\Gamma)$.

Now, let $f : I \to I'$ and $f' : I' \to I''$ be comorphism of institutions. $F(f' \bullet f) = \langle \phi' \circ \phi, \alpha' \bullet \alpha \rangle = F(f') \bullet F(f)$ and $F(Id_I) = Id_{F(I)}$. Then F is a functor.

Consider now the application:

$$\begin{array}{rccc} G: & \pi - \mathbf{Inst} & \longrightarrow & \mathbf{Inst} \\ & J & \longmapsto & G(J) = \langle \mathbb{S}ig, Sen, Mod^J, \models^J \rangle \end{array}$$

where:

- The two first components of the $\pi-$institution are preserved.
- $Mod^J : \mathbb{S}ig \to \mathbb{C}at^{op}$.

$Mod^J(\Sigma) := \{C_\Sigma(\Gamma);\ \Gamma \subseteq Sen(\Sigma)\} \subseteq P(Sen(\Sigma))$ is viewed as a poset category and, given $f : \Sigma \to \Sigma'$, $Mod^J(f) = Sen(f)^{-1}$.
$Mod^J(f)$ is well defined. Indeed: Let $\Gamma \subseteq Sen(\Sigma')$ and $\varphi \in C_\Sigma(Sen(f)^{-1}(C_{\Sigma'}(\Gamma)))$.

$$\begin{array}{rcl} Sen(f)(\varphi) \in Sen(f)[C_\Sigma(Sen(f)^{-1}(C_{\Sigma'}[\Gamma]))] & \subseteq & C_\Sigma[Sen(f)(Sen(f)^{-1}(C_\Sigma[\Gamma]))] \\ & \subseteq & C_{\Sigma'}(C_{\Sigma'}[\Gamma]) = C_{\Sigma'}[\Gamma]. \end{array}$$

Therefore $\varphi \in Sen(f)^{-1}(C_\Sigma[\Gamma])$. It is easy to see that Mod^J is a contravariant functor.

• Define $\models^J \subseteq |Mod(\Sigma)| \times Sen(\Sigma)$ as a relation such that, given $m \in Mod(\Sigma)$ and $\varphi \in Sen(\Sigma)$, $m \models^J_\Sigma \varphi$ if and only if $\varphi \in m$. Let $f : \Sigma \to \Sigma'$, $\varphi \in Sen(\Sigma)$ and $m' \in |Mod(\Sigma')|$.

$$\begin{array}{rcl} Mod^J(f)(m') \models^J_\Sigma \varphi & \Leftrightarrow & Sen(f)^{-1}(m') \models^J_\Sigma \varphi \\ & \Leftrightarrow & \varphi \in Sen(f)^{-1}(m') \\ & \Leftrightarrow & Sen(f)(\varphi) \in m' \\ & \Leftrightarrow & m' \models^J_{\Sigma'} Sen(f)(\varphi). \end{array}$$

Therefore the compatibility condition is satisfied and then we have that $G(J)$ is an institution.

Now, let $h = \langle \phi, \alpha \rangle : J \to J'$ be a comorphism of π-institutions. Define, for any $\Sigma \in |\mathbb{S}ig|$, $\beta_\Sigma : Mod^{J'} \circ \phi(\Sigma) \to Mod^J(\Sigma)$ where $\beta_\Sigma(m) = \alpha_\Sigma^{-1}(m)$. We prove that β_Σ is well defined, i.e., $\alpha_\Sigma^{-1}(m) \in Mod^J(\Sigma)$. Let $\varphi \in C_\Sigma(\alpha_\Sigma^{-1}(m))$. Since h is a morphism of π-institutions, then $\alpha_\Sigma(\varphi) \in C_{\phi(\Sigma)}(\alpha_\Sigma(\alpha_\Sigma^{-1}(m))) \subseteq C_{\phi(\Sigma)}(m) = m$. Therefore $\varphi \in \alpha_\Sigma^{-1}(m)$.

Now we prove that β is a natural transformation. Let $f : \Sigma_1 \to \Sigma_2$. Since α is a natural transformation, the following diagram commutes:

$$\begin{array}{ccc} P(Sen(\Sigma_1)) & \xleftarrow{\alpha_{\Sigma_1}^{-1}} & P(Sen'(\phi(\Sigma_1))) \\ \uparrow {\scriptstyle Sen(f)^{-1}} & & \uparrow {\scriptstyle Sen'(\phi(f))^{-1}} \\ P(Sen(\Sigma_2)) & \xleftarrow[\alpha_{\Sigma_2}^{-1}]{} & P(Sen'(\phi(\Sigma_2))) \end{array}$$

Using this commutative diagram we are able to prove that the following diagram commutes:

$$\begin{array}{ccc} Mod^{J'} \circ \phi(\Sigma_1) & \xrightarrow{\beta_{\Sigma_1}} & Mod^J(\Sigma_1) \\ \uparrow {\scriptstyle Mod^{J'}(\phi(f))} & & \uparrow {\scriptstyle Mod^J(f)} \\ Mod^{J'} \circ \phi(\Sigma_2) & \xrightarrow[\beta_{\Sigma_2}]{} & Mod^J(\Sigma_2) \end{array}$$

Let $m \in Mod^{J'} \circ \phi(\Sigma_2)$.

$$\begin{array}{rcl} Mod^J(f) \circ \beta_{\Sigma_2}(m) & = & Mod^J(f)(\alpha_{\Sigma_2}^{-1}(m)) \\ & = & Sen(f)^{-1}(\alpha_{\Sigma_2}^{-1}(m)) \\ & = & \alpha_{\Sigma_1}^{-1}(Sen(\phi(f))^{-1}(m)) \\ & = & \beta_{\Sigma_1}(Sen(\phi(f))^{-1}(m)) \\ & = & \beta_{\Sigma_1} \circ Mod^{J'}(\phi(f))(m). \end{array}$$

$G(h) = \langle \phi, \alpha, \beta \rangle$ is a comorphism of institutions. Indeed, it is enough to prove the compatibility condition. Let $m \in Mod^{J'}(\phi(\Sigma))$ and $\varphi \in Sen(\Sigma)$.

$$\begin{array}{rcl} m \models^{J'}_{\phi(\Sigma)} \varphi\alpha_\Sigma(\varphi) & \Leftrightarrow & \alpha_\Sigma(\varphi) \in m \\ & \Leftrightarrow & \varphi \in \alpha_\Sigma^{-1}(m) \\ & \Leftrightarrow & \varphi \in \beta_\Sigma(m) \\ & \Leftrightarrow & \beta_\Sigma(m) \models^J_\Sigma (m)\varphi. \end{array}$$

It is easy to see that G is a functor.

Theorem 2.9 *The functors* $F : \mathbf{Inst} \to \pi - \mathbf{Inst}$ *and* $G : \pi - \mathbf{Inst} \to \mathbf{Inst}$ *defined above establish an adjunction* $G \dashv F$ *between the categories* **Inst** *and* $\pi - \mathbf{Inst}$.

Proof.

Define the application $\eta_J = \langle Id_{\mathbb{S}ig}, Id_{Sen}\rangle : J \to F(G(J))$ for each π-institution $J = \langle \mathbb{S}ig, Sen, \{C_\Sigma\}_{\Sigma\in|\mathbb{S}ig|}\rangle$. This application is well defined. Indeed, we prove that $C_\Sigma = C_\Sigma^{G(I)}$ for any $\Sigma \in |\mathbb{S}ig|$. By definition of the functor G, notice that given $\Sigma \in |\mathbb{S}ig|$ and $\Gamma \subseteq Sen(\Sigma)$, $C_\Sigma(\Gamma) \in \Gamma^\star = \{m \in Mod(\Sigma) : m \models^J_\Sigma \Gamma\}$. Moreover $C_\Sigma(\Gamma) \subseteq m$ for every $m \in \Gamma^\star$. Then for any $\varphi \in Sen(\Sigma)$

$$\begin{array}{rcl} \varphi \in C_\Sigma(\Gamma) & \Leftrightarrow & \varphi \in m \ for\ all\ m \in \Gamma^\star \\ & \Leftrightarrow & m \models^J_\Sigma \varphi \ for\ all\ m \in \Gamma^\star \\ & \Leftrightarrow & \varphi \in \Gamma^{\star\star} = \{\psi \in Sen(\Sigma) \ : \ \Gamma^\star \models^J_\Sigma \psi\} \\ & \Leftrightarrow & \varphi \in C_\Sigma^{G(J)}(\Gamma). \end{array}$$

It is clear that $(\eta_J)_{J\in|\pi-\mathbf{Inst}|}$ is a natural transformation. It remains to prove that η_J satisfies the universal property for any $J \in |\pi - \mathbf{Inst}|$.

Let $h = \langle \phi, \alpha\rangle : J \to F(I)$ where $J = \langle \mathbb{S}ig, Sen, \{C_\Sigma\}_{\Sigma\in|\mathbb{S}ig|}\rangle$ is a $\pi-$institution, $I = \langle \mathbb{S}ig', Sen', Mod', \models'\rangle$ an institution and h a morphism of $\pi-$institutions. Define $\bar{h} = \langle \phi, \alpha, \beta\rangle : G(J) \to I$ where the first two components are the same of h and, given $\Sigma \in |\mathbb{S}ig|$, $\beta_\Sigma : Mod' \circ \phi(\Sigma) \to Mod^J(\Sigma)$ such that $\beta_\Sigma(m) = \alpha_\Sigma^{-1}[m^\star]$. β_Σ is well-defined. Indeed, notice that $m^\star = m^{\star\star\star}$ for any $m \in Mod'(\phi(\Sigma))$. Since $C^I_\Sigma(\Gamma) = \Gamma^{\star\star}$, therefore $m^\star = C^I_\Sigma(m^\star)$. We have shown that as h is a morphism of $\pi-$institutions, $\alpha_\Sigma^{-1}(m^\star) = \alpha_\Sigma^{-1}(C^I_\Sigma(m^\star)) \in Mod^J$.

Now we prove that $(\beta_\Sigma)_{\Sigma\in|\mathbb{S}ig|}$ is a natural transformation. Let $f : \Sigma_1 \to \Sigma_2$. Then given $m \in Mod' \circ \phi(\Sigma_2)$

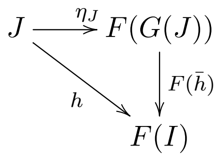
$$\begin{array}{ccc} Mod' \circ \phi(\Sigma_1) & \xrightarrow{\beta_{\Sigma_1}} & Mod^J(\Sigma_1) \\ {\scriptstyle Mod'(\phi(f))}\uparrow & & \uparrow{\scriptstyle Mod^J(f)} \\ Mod' \circ \phi(\Sigma_2) & \xrightarrow[\beta_{\Sigma_2}]{} & Mod^J(\Sigma_2) \end{array}$$

$$\begin{array}{rcl} Mod^J(f)(\beta_{\Sigma_2}(m)) & = & Sen(f)^{-1}(\alpha_{\Sigma_2}^{-1}(m^\star)) \\ & = & \alpha_{\Sigma_1}^{-1}(Sen(\phi(f)^{-1})(m^\star)) \\ & =_\dagger & \alpha_{\Sigma_1}^{-1}((Mod(\phi(f))(m^\star))^\star) \\ & = & \beta_{\Sigma_1}(Mod(\phi(f))(m)). \end{array}$$

The justification of the equality (†) is:

$$\begin{aligned}
\varphi \in Sen(\phi(f))^{-1}(m^\star) &\Leftrightarrow Sen(\phi(f))(\varphi) \in m^\star \\
&\Leftrightarrow m \models_{\phi(\Sigma_2)} Sen(\phi(f))(\varphi) \\
&\Leftrightarrow Mod(\phi(f))(m) \models \Sigma_2 \varphi \\
&\Leftrightarrow \varphi \in (Mod(\phi(f))(m))^\star
\end{aligned}$$

hence β is a natural transformation. Therefore $\bar{h}$ is a comorphism between $G(I)$ and I. Observe that $F(\bar{h}) = \langle \phi, \alpha \rangle = h$. Then we have the following diagram commuting:



Moreover, clearly $\bar{h}$ is the unique arrow such that the diagram above commutes. Hence $G \dashv F$. ■

Remark 2.10 *Note that $F \circ G = Id_{\pi-\mathbf{Inst}}$ and the unity of this adjunction, the natural transformation $\eta : Id_{\pi-\mathbf{Inst}} \to F \circ G$, is the identity. Thus, the category $\pi - \mathbf{Inst}$ can be seen as a full co-reflective subcategory of* **Inst**.

3 Final remarks and future work

Remark 3.1 *In [**MaMe**] it is presented a right adjoint $(-)_L : \mathcal{L}_f \to \mathcal{L}_s$ to the "inclusion" functor $(+)_L : \mathcal{L}_s \to \mathcal{L}_f$ (see Example 2.7). It will be interesting to understand the role of these adjoint pairs of functors between the logical categories $(\mathcal{L}_s, \mathcal{L}_s)$ at the π-institutional level (J_f, J_s).*

Remark 3.2 *The "proof-theoretical" Example 2.7, that provides* π-institutions *(J_f, J_s) for categories of propositional logics $(\mathcal{L}_s, \mathcal{L}_s)$, leads us to search for an analogous "model-theoretical" version of it that is different from the canonical one (i.e., that obtained by applying the functor $G : \pi - \mathbf{Inst} \to \mathbf{Inst}$). In [**MaPi2**], we provide (another)* institutions *for each category of propositional logics, through the use of the notion of a* matrix *for a propositional logic. Moreover, by a convenient modification of the later construction, we provide in [**MaPi2**] an institution for each "equivalence class" of algebraizable logic: this enables us to apply notions and results from Institution Theory in the propositional logic setting and derive, from the introduction of the notion of "Glivenko's context", a strong and general form of Glivenko's Theorem relating two "well-behaved" logics.*

The examination of the content mentioned in the remarks above could lead naturally to consider new categories of propositional logics and to a new notions of morphism of (π-)institutions.

This work also opens a way to investigate categorial properties of the categories of institutions and π-institutions with many kinds of morphisms in each of them.

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