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Model-theoretic applications of the profinite hull functor of special groups

Hugo Luiz Mariano and Francisco Miraglia

Abstract

This work is a development of some model-theoretic aspects of the theory of Special Groups ([9]) − a first order axiomatization of the algebraic theory of quadratic forms. In [33] we constructed a functor, the profinite hull functor, $\mathcal{P}$, from the category RSG of reduced special groups with SG-morphisms to the category RSG_{pf} of profinite reduced special groups with continuous SG-morphisms. From results concerning $\mathcal{P}$, we obtain here an alternative and simple proof that the class of reduced special satisfying an interesting local-global property - for p.p. formulas ([27], [28]) - is an elementary class in the language of special groups, axiomatizable by Horn sentences or by $\forall\exists$-sentences, a result originally established in [6].

Keywords: special groups, profinite hull, local-global principles, positive-existential formulas.

Introduction

Special Groups (SGs) are a first-order axiomatization of the Algebraic Theory of Quadratic Forms (ATQF). The standard ATQF study classes of isometry of finite-dimensional vector spaces endowed with a bilinear symmetric form, defined over a field F with $char(F) \neq 2$. Equivalently, the theory concerns classes of isometry of $n \times n$ symmetric matrix over F, for each $n \in \mathbb{N}$, in particular, for any such matrix S, there are a $n \times n$ matrix T and $a_1, \cdots, a_n \in F$ such that $S = T^t Diag(a_1, \ldots, a_n)T$. It is a fundamental result that "binary isometry determines n-ary isometry", $n \in \mathbb{N}$. In 1937, E. Witt introduced the ring $W(F)$, that classifies the classes of isometry of non-singular and anisotropic quadratic forms over F. The coefficients of non-singular diagonal quadratic form are all in $F^\bullet := F \setminus \{0\}$: the traditional theory of quadratic forms over F have focus on the exponent 2 group $F^\bullet/F^{\bullet 2}$ and the so called *reduced* *theory* have coefficients in the group $F^\bullet/(\sum F^2)^\bullet$. The main reference on ATQF is the book [21].

The suitable first-order language for the study of Special Groups, L_{SG}, contains two symbols for constants (1 and -1), one symbol for binary operation (multiplication) and one symbol for quaternary relation ($\equiv$, the isometry between quadratic forms with dimension two). The class of special groups, **SG**, (respectively, the class of **reduced** special groups **RSG**) is axiomatizable by L_{SG}-sentences of the form $\forall\vec{x}(\psi_0(\vec{x}) \to \psi_1(\vec{x}))$ where $\psi_i(\vec{x})$ is positive existential (p.e.), (respectively, by sentences that are either the negation of an atomic sentence of the form $\forall\vec{x}(\psi_0(\vec{x}) \to \psi_1(\vec{x}))$, with ψ_i p.e.). The central reference for special groups (SGs) is [9].

Section 1 below presents the definitions and results needed in the sequel. The theory of quadratic forms over fields is encoded into the theory of special groups via a covariant functor from the category of fields (with characteristic $\neq 2$) into the category of special groups; formally real Pythagorean fields are mapped by this functor into the category of reduced special groups (RSGs). A non first-order abstract presentation of the (reduced) algebraic theory of quadratic forms is given by Murray Marshall's notion of *space of orderings*. The main reference on this subject is [25]. In Chapter 3 of [9] it is shown that there is a *duality* between the category of reduced special groups and the category of spaces of orderings.

Local-global principles are a fundamental subject in (abstract) presentations of ATQF. Many of them have a functorial encoding in the theory of SGs via "hulls functor": an example is the Boolean Hull Functor, B ([9], [10], [12]) and the Profinite Hull Functor $\mathcal{P}$ ([32], [33], [34], [35]). We present a brief reminder on local-global principles in Section 2. In Section 3, we use the functorial encoding by $\mathcal{P}$ of a local-global principle of a logical nature ([27], [28]) $-$ with respect to positive primitive L_{SG}-formulas $-$ to provide an alternative and simple proof that the class of reduced special satisfying this local-global property is an elementary class in the language L_{SG}, axiomatizable by sets of Horn sentences or by a set of $\forall\exists$-sentences, a result due to [6].

1 Preliminaries

1.1 Special Groups (SG)

Many concepts in the theory of SGs can be described by positive-existential L_{SG}-formulas. Examples include the isometry between n-forms, the notion of isotropic form and the relation of a form being a subform of another. The reader is referred to [9] for all undefined notions used below (e.g., pre-special group).

1 Basic Notions. Let G be a special group and let $n \geq 1$ be an integer.
• A ***n*-form over *G***, $\varphi = \langle a_1, \ldots, a_n \rangle$, is an element G^n; the integer n is called the **dimension** of φ. There are natural definitions of n-dimensional isometry, of sum ($\oplus$),

product ($\otimes$), isotropy, anisotropy and hiperbolicity of forms over G. Furthermore, sum and product of forms are associative and commutative.

• If φ, ψ are forms over G, φ is a **subform** ψ (notation: $\varphi \preceq_G \psi$) if there is a form θ such that $\psi \equiv_G \varphi \oplus \theta$; by Witt cancellation law (Proposition 1.6.(b) in [9]), θ is unique up to isometry.

• ψ is a form over G, the **Witt index** of ψ (notation: $\text{ind}_W(\psi)$) is 0 if ψ is anisotropic; otherwise, it is the largest integer $k \geq 1$ such that $k\langle 1, -1 \rangle \preceq_G \psi$; hence, $\psi \equiv_G \text{ind}_W(\psi)\langle 1, -1 \rangle \oplus \psi_{an}$, where ψ_{an} is the **anisotropic part** of ψ, which is unique up to isometry.

• An element $a \in G$ is **represented by** a form ψ, written $a \in D_G(\psi)$, if $\langle a \rangle \preceq_G \psi$. □

2 Pfister forms. A **Pfister form** of degree $k \geq 1$ over G is a form $P = \langle 1, a_1 \rangle \otimes \ldots \otimes \langle 1, a_k \rangle$, for some $a_1, \ldots, a_k \in G$. If P is a Pfister form over G, then $D_G(P) = \{a \in G : aP \equiv_G P\}$ and is a subgroup of G. We register that any isotropic Pfister form must be hyperbolic. □

3 The representation relation. Sometimes it is convenient to have in our first-order language of special groups a unary predicate, $D(1, \cdot)$, to express the representation by binary forms of the type $\langle 1, a \rangle$. This language is interdefinable with L_{SG}, modulo the axioms of *pre-special groups*. Hence, we may assume that a L_{SG}-atomic formula $\phi(\vec{v})$ can be written as $t_0(\vec{v}) = t_1(\vec{v})$ or $t_0(\vec{v}) \in D(1, t_1(\vec{v}))$, for some L_{SG}-terms $t_i(\vec{v})$. If φ, ψ are forms of dimension $n \in \mathbb{N}$ over a pre-special group G, it follows straightforwardly from the definition of isometry of n-forms, that $\varphi \equiv_G \psi$ can be described by a positive primitive (pp) L_{SG}-formula, whose parameters are the entries of φ, ψ. Analogously, we can describe the following notions by means of pp-sentences of L_{SG} with parameters in the entries of the forms involved:

• "φ is a Pfister form";
• "φ is an isotropic form";
• "$\text{ind}_W(\varphi) \geq k$", for fixed $k \in \mathbb{N}$;
• "φ is subform of ψ", i.e. $\varphi \preceq \psi$. □

1.2 SG-morphisms

Special group morphisms (SG-morphisms) are simply the L_{SG}-morphisms of the underlying L_{SG}-structures. If $f : G \longrightarrow H$ is a SG-morphism and $\varphi = \langle a_1, \ldots, a_n \rangle$ is a n-form over G, write $f \star \varphi = \langle f(a_1), \ldots, f(a_n) \rangle$ for the f-image form over H. In Chapter 5 of [9] there is a detailed study of several types of morphisms between L_{SG}-structures.

4 Monomorphisms. Some special kinds of *monomorphisms* appear naturally in the category **SG**:

• complete embeddings, i.e., the SG-morphisms that preserve and reflect isometry of n-forms, $n \geq 1$;

- SG-morphisms that preserve and reflect isotropy;
- SG-morphisms that preserve and reflect subforms;
- Elementary (resp., pure) SG-morphisms, i.e., those L_{SG}-morphisms that preserve and reflect arbitrary (resp., positive-existential) L_{SG}-formulas;
- L_{SG}-sections, i.e., the SG-morphisms, $G \xrightarrow{s} H$, such that there is a SG-morphism, $H \xrightarrow{r} G$, satisfying $r \circ s = Id_G$. □

5 Epimorphisms. The most useful notion of SG-epimorphism is that of the projection of a reduced special group (RSG) on a quotient by a *saturated* subgroup; the (proper) saturated subgroups classify the congruences on a RSG whose associated quotient is a RSG. If G is a RSG and Σ is a saturated subgroup of G, write $p_\Sigma : G \longrightarrow G/\Sigma$ for the canonical SG-quotient morphism. If $\varphi = \langle a_1, \ldots, a_n \rangle$ is a n-form over G, whenever convenient, we write $\varphi/\Sigma = \langle a_1/\Sigma, \ldots, a_n/\Sigma \rangle = p_\Sigma \star \varphi$ for the image form over G/Σ. □

6 Classes of saturated subgroups. If G is a SG,
- $Ssat(G)$ is the set of saturated subgroups of G;
- $Ssat^\star(G)$ is the set of *proper* saturated subgroups of G;
- $X(G)$ is the set of *maximal* saturated subgroups of G;
- $\mathcal{F}(G)$ is the set of saturated subgroups of G of *finite index* in G. □

Remark 1.1 *The maximal saturated subgroups of G are precisely the kernels of the* **SG**-*morphisms $G \to \mathbb{Z}_2$, i.e. the kernels of the elements of $X_G = Hom_{SG}(G, \mathbb{Z}_2)$, called the* space of orderings *of G; moreover, this association is* bijective. □

1.3 Boolean algebras and the Boolean Hull Functor of a RSG

In Chapters 4, 5 and 7 in [9] there is an extensive analysis of the interaction between Boolean algebras and special groups. In particular, the Boolean hull functor and its properties are an essential tool in the solutions of many questions in quadratic form theory (see [10], [11], [12]). We provide here just the definitions and the results needed below, referring the reader to the above references.

7 Boolean Algebras as Special Groups

- Let $\langle B, \vee, \wedge, \perp, \top \rangle$ be a Boolean algebra (BA). Then $\langle B, \vartriangle, \perp \rangle$, where $\vartriangle$ is symmetric difference, is a group of exponent 2 and $\langle B, \vartriangle, \wedge, \perp, \top \rangle$ is a Boolean unitary ring.

For each a, b, c, $d \in B$, define

[$\equiv_B$] $\quad \langle a, b \rangle \equiv_B \langle c, d \rangle \quad \Leftrightarrow \quad a \wedge b = c \wedge d \;$ and $\; a \vee b = c \vee d$.

- By Corollary 4.4.(b) in [9], $\langle \langle B, \vartriangle, \perp \rangle, \equiv_B, \top \rangle$ is RSG, where $1 := \perp$, $-1 := \top$. □

8 BAs and RSGs

• The following table describes the correspondence between BA concepts and RSG concepts:

Reduced special groups	Boolean algebras
$\cdot$	$\triangle$
1	$\bot$
-1	$\top$
$a \in D_G(1, b)$	$a \leq b$
Saturated subgroup	Ideal
RSG-morphism	BA-morphism

• Let G be a RSG. Then G is (associated to) a Boolean algebra iff for each $a, b \in G$, the form $\ll 1, a, b, -ab\rangle$ is isotropic (Proposition 7.17 in [9]).

• Let **BA** be the category of Boolean algebras and **BA**-morphisms. We have a functor, $\gamma : \mathbf{BA} \longrightarrow \mathbf{RSG}$, identifying **BA** with a *full* subcategory of **RSG**, justifying its frequent omission from the notation. □

9 The Boolean Hull of RSGs

• Let G be a reduced special group, $X_G = Hom_{SG}(G, \mathbb{Z}_2)$ be its space of orders and let $B_G := \mathcal{B}(X_G)$ be BA of clopen subsets of X_G. The map $\varepsilon_G : G \longrightarrow B_G$, given by

$$\varepsilon_G(a) = [a = -1] = \{\sigma \in X_G : \sigma(a) = -1\} = [-a = 1],$$

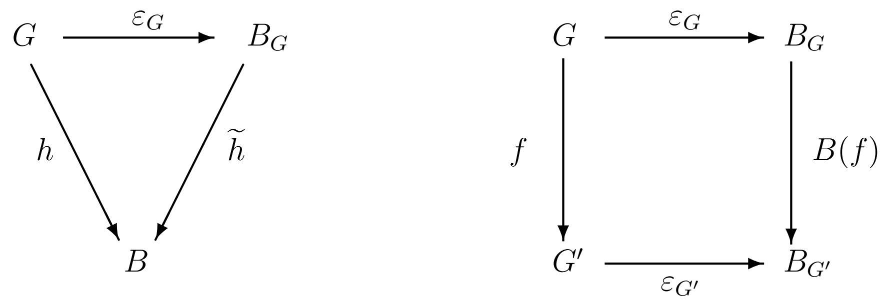
is a RSG-embedding and the diagram $G \stackrel{\varepsilon_G}{\longrightarrow} B_G$ is the **Boolean hull of G**.

• If $G \stackrel{f}{\to} \longrightarrow G'$ is a RSG-morphism, let $X_G \stackrel{X(f)}{\to}\longleftarrow X_{G'}$ be the induced continuous map. Now let $B_G \stackrel{B(f)}{\to} \longrightarrow B_{G'}$ be the BA-morphism dual to $X(f)$, given by $B(f)(U) := X(f)^{-1}[U]$, $U \in \mathcal{B}(X_G)$. □

10 Properties of the Boolean Hull

• The map $(G \stackrel{f}{\to} \longrightarrow G') \mapsto (B_G \stackrel{B(f)}{\to} \longrightarrow B_{G'})$, is a covariant functor, $B : \mathbf{RSG} \longrightarrow \mathbf{BA}$, the **Boolean Hull Functor**.

• The family $\{(G \stackrel{\varepsilon_G}{\to} \longrightarrow B_G) : G \in Obj(\mathbf{RSG})\}$ is a natural transformation, $\varepsilon : Id_{\mathbf{RSG}} \longrightarrow \gamma \circ B$, i.e., the square below is commutative, for all RSG-morphism $f : G \longrightarrow G'$.



• If B is a BA and $h : G \longrightarrow B$ is a SG-morphism, there is a unique BA-morphism, $\widetilde{h} : B_G \longrightarrow B$, such that the triangle above is commutative.

• The Boolean hull functor is left adjoint to the "inclusion" functor $\gamma : \mathbf{BA} \longrightarrow \mathbf{RSG}$ and the natural transformation $\varepsilon : Id_{\mathbf{RSG}} \longrightarrow \gamma \circ B$ is the unit of this adjunction.

• Since the "inclusion", $\gamma : \mathbf{BA} \longrightarrow \mathbf{RSG}$ is a right adjoint it preserves all limits. Hence, when we consider limits of BAs, we need not specify the limit as a BA or a RSG. □

1.4 Profinite RSGs and the Profinite Hull Functor of a RSG

Profinite RSGs appeared (in the dual setting of orderings spaces) in [23] and were studied in [22]; [31] and [32] describe logical-categorical properties of profinite structures and the profinite hull functor in a general setting; [36], [33], [34] and [35] describe properties and applications of the notions of profiniteness and of the profinite hull in the context of special groups, while [5] establishes a representation of profinite RSGs by Pythagorean fields. As above, we register the definitions and the results needed in what follows, referring the reader to the aforementioned references.

11 Logical notions and profinite structures

(I) Recall that a formula in L is:

• **positive existential (p.e.)** if it is equivalent to a formula constructed from the atomic formula employing only the connectives $\wedge$, $\vee$ and the existential quantifier $\exists$;

• **positive primitive (p.p.)** if it is equivalent to a formula of the form $\exists\, \overline{x}\, \phi$, where ϕ is a conjunction of atomic formulas;

• **geometrical** if it is logically equivalent to one of the form $\forall\, \overline{x}(\phi(\overline{x}, \overline{y}) \rightarrow \psi(\overline{x}, \overline{y}))$, where ϕ, ψ are p.e.-formulas, **or** to the negation of an atomic formula;

• **basic Horn formula** is a disjunction $\phi_1 \vee \ldots \vee \phi_k$ where at most one of the formulas ϕ_i is an atomic formula and all the others are negations of atomic formulas;

• **Horn formula** if it is build up from basic Horn formulas by the use of $\wedge$, $\exists$, $\forall$.

It is well-known that every p.e.-formula is equivalent to the disjunction of finite conjunctions of p.p.-formulas.

(II) A map between L-structures, $f : M \longrightarrow N$, is a **pure L-morphism** if for each p.e.-formula $\varphi(\vec{x})$ and for all $\overline{a}$ in M, $M \models \varphi[\overline{a}] \;\Leftrightarrow\; N \models \varphi[f\overline{a}]$. Hence, a L-morphism $g : M \longrightarrow N$ is pure iff it reflects the satisfaction of p.p.-formulas.

Clearly, all pure L-morphisms are L-embeddings and any elementary embedding and any L-section are pure L-morphisms. Moreover, if Σ is a set of geometrical L-sentences and $f : M \longrightarrow N$ is a pure L-morphism, then $N \models \Sigma \;\Rightarrow\; M \models \Sigma$.

(III) Let L be a first-order language and let *L-mod* be the category of L-structures and L-morphisms.

- A **downward directed poset** is a poset $\langle I, \leq \rangle$ that is *non-empty* and such that, for each $i, j \in I$, there is $k \in I$ with $k \leq i, j$.
- A **cofiltered system** of L-structures is a functor (or diagram), $\mathcal{D}$, from a downward directed poset $\langle I, \leq \rangle$ into the category L-mod.
- An L-structure is **profinite** if it is L-isomorphic to the limit, in the category L-mod, of a *cofiltered system* of *finite* L-structures.

(IV) Let $\langle I, \leq \rangle$ be a downward directed poset and let $\mathcal{M} = \ll \ (M_i \overset{f_{ij}}{\to} \longrightarrow M_j) : (i \leq j) \in I \ \rangle$ be a cofiltered system of L-structures and L-morphism over I.

- Denote $\Pi(\mathcal{M}) := \prod_{i \in I} M_i$, the product L-structure and, for each $i \in I$, let $\pi_i : \Pi(\mathcal{M}) \longrightarrow M_i$ be the corresponding coordinate projection.
- The (essentially unique) limit of $\mathcal{M}$ is a commutative cone $\{(P \overset{\mathfrak{p}_i}{\to} \longrightarrow M_i) : i \in I\}$ over the diagram $\mathcal{M}$, where P is a L-structure and $\mathfrak{p}_i : P \longrightarrow M_i$ are L-morphisms satisfying a well-known universal property.
- There is a natural L-monomorphism, $\iota : P \longrightarrow \Pi(\mathcal{M})$, given by $\iota(x) = \langle \mathfrak{p}_i(x) \rangle_{i \in I}$, such that for all i and all $i \leq j$ in I: $f_{ij} \circ \mathfrak{p}_i = \mathfrak{p}_j$ and $\pi_i \circ \iota = \mathfrak{p}_i$.
- The limit P may (and often is) identified with the *closed*[1] L-substructure $\iota[P]$ of M given by:

$$\iota[P] = \{\langle a_i \rangle_{i \in I} \in \Pi(\mathcal{M}) : \forall\, i \leq j \text{ in } I,\ \ f_{ij}(a_i) = a_j\}.$$

Under this identification, the L-morphism $\mathfrak{p}_i$ is the restriction to $\iota[P]$ of the projection π_i, $i \in I$.

(V) A very useful characterization of profinite SGs comes from the following general result in [30][2]:

Profinite L-structures are retracts of ultraproducts of finite L-structures.

More precisely, if P is the limit of a cofiltered system $\mathcal{M} = \ll \ (M_i \overset{f_{ij}}{\to} \longrightarrow M_j) : (i \leq j) \in I \ \rangle$ of finite L-structures over the downward directed poset $\langle I, \leq \rangle$, there is an ultrafilter U over I together with L-morphisms,

$$P \overset{\iota}{\longrightarrow} \prod_{i \in I} M_i \overset{q}{\longrightarrow} \prod_{i \in I} M_i/U \overset{\gamma^U}{\longrightarrow} P$$

[1] I.e. $\iota[P]$ is a closed subset of $\prod_{i \in I} M_i$ endowed with the *product* (Boolean) topology.

[2] Generalizing to the category L-mod Lemma 4.4 in [23], stated for spaces of orderings.

such that $\gamma^U \circ q \circ \iota \ = \ Id_P$, where ι is the canonical embedding of P into the product of the M_i and q is the natural quotient morphism.
• If T is a geometrical L-theory, then Mod(T), the full subcategory of *L-mod* consisting of models of T, is closed under profinite limits. In particular, as **RSG** is a L_{SG}-elementary class axiomatizable by geometric sentences, then the inclusion functor **RSG** $\hookrightarrow$ L_{SG}-mod creates profinite limits. □

12 The class of profinite RSGs

• In [22] the reader will find a *topological characterization* of the profinite RSGs, analogous to the well-known description of profinite topological groups and spaces. A general statement for topological profinite structures in certain elementary classes in *L-mod* is presented in [32].
• The topological characterization of profinite RSGs yields closure properties of $\mathbf{RSG}_{pf} \subseteq$ **RSG**. In particular, a non-empty product of profinite RSGs is a profinite RSG. □

13 The Profinite Hull of RSGs

• Let G be a reduced special group and let $\mathcal{F}(G)$ be the set of all saturated subgroup of finite index in G. Since $\mathcal{F}(G)$ is the closure of $X(G)$ under finite intersections, it is downward directed by inclusion (in fact, it is a filter in the complete algebraic lattice of saturated subgroups of G).
• For $\Gamma \subseteq \Delta$ in $\mathcal{F}(G)$, let $p_{\Gamma\Delta} : G/\Gamma \longrightarrow G/\Delta$ be the unique SG-morphism such that $p_{\Gamma\Delta} \circ p_\Gamma \ = \ p_\Delta$, where p_Δ and p_Γ are the canonical quotient SG-morphisms. Thus,

$$\mathcal{G} \ = \ \ll \ (G/\Gamma \overset{p_{\Gamma\Delta}}{\to}\!\!\twoheadrightarrow \ G/\Delta) : (\Gamma \subseteq \Delta) \in \mathcal{F}(G) \ \rangle$$

is a cofiltered system of finite RSGs over the downward directed poset $(\mathcal{F}(G), \subseteq)$.
• If $\mathcal{P}(G) = \varprojlim_{\Delta \in \mathcal{F}(G)} G/\Delta$, then $\{(\mathcal{P}(G) \overset{\mathfrak{p}_\Delta}{\to} \longrightarrow G/\Delta) : \Delta \in \mathcal{F}(G)\}$ is a limit cone over $\mathcal{G}$.
• We know that $\mathcal{P}(G) \in \mathbf{RSG}_{pf}$ and if $\Pi(G) := \prod_{\Delta \in \mathcal{F}(G)} G/\Delta$, then we also have $\Pi(G) \in \mathbf{RSG}_{pf}$ (endowed with the product topology). To keep notation straight, let $\iota : \mathcal{P}(G) \hookrightarrow \Pi(G)$ be the inclusion morphism and consider the map $a \in G \longmapsto \langle\, a/\Delta \,\rangle_{\Delta \in \mathcal{F}(G)}$: it will be written as δ_G if its codomain is $\Pi(G)$ and as η_G if it is considered as a map into $\mathcal{P}(G)$:

$$\delta_G : G \longrightarrow \Pi(G) \quad \text{and} \quad \eta_G : G \longrightarrow \mathcal{P}(G).$$

and so $\delta_G = \iota \circ \eta_G$.
• If $G \overset{f}{\to} \longrightarrow G'$ is a *RSG*-morphism, then $f^* : \mathcal{F}(G') \to \mathcal{F}(G)$, given by $\Delta' \mapsto f^{-1}[\Delta']$, is a well-defined and increasing function of downward directed posets. Moreover, let $f_{\Delta'} : G/f^*(\Delta') \rightarrowtail G'/\Delta'$, given by $g/f^{-1}[\Delta'] \mapsto f(g)/\Delta'$, be the *injective RSG*-morphism induced on quotients. Then, the family $\{(\mathcal{P}(G) \overset{f_{\Delta'} \circ \mathfrak{p}_{f^*(\Delta')}}{\to} \longrightarrow G'/\Delta') : \Delta' \in \mathcal{F}(G')\}$ is a commutative cone over the diagram

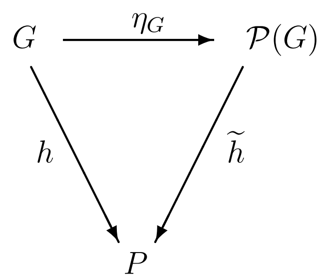
$$\mathcal{G}' = \ll (G'/\Gamma' \overset{p_{\Gamma'\Delta'}}{\to} \longrightarrow G'/\Delta') : (\Gamma' \subseteq \Delta') \in \mathcal{F}(G') \rangle.$$

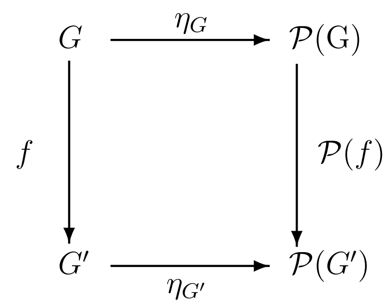
Thus, by the universal property of the limit cone $\{(\mathcal{P}(G') \overset{\mathfrak{p}'_{f^*(\Delta')}}{\to} \longrightarrow G'/\Delta') : \Delta' \in \mathcal{F}(G')\}$, there is a unique *continuous* RSG-morphism, $\mathcal{P}(f) : \mathcal{P}(G) \to \mathcal{P}(G')$, such that $\mathfrak{p}'_{\Delta'} \circ \mathcal{P}(f) = f_{\Delta'} \circ \mathfrak{p}_{f^*(\Delta')}$, for each $\Delta' \in \mathcal{F}(G')$. □

14 Properties of the Profinite Hull

• The map $(G \overset{f}{\to} \longrightarrow G') \mapsto (\mathcal{P}(G) \overset{\mathcal{P}(f)}{\to} \longrightarrow \mathcal{P}(G'))$, is a covariant functor, $\mathcal{P} : \mathbf{RSG} \longrightarrow \mathbf{RSG}_{pf}$, called the **Profinite Hull Functor of RSGs**.

• The family $\{(G \overset{\eta_G}{\to} \longrightarrow \mathcal{P}(G)) : G \in Obj(\mathbf{RSG})\}$ is a natural transformation, $\eta : Id_{\mathbf{RSG}} \longrightarrow U \circ \mathcal{P}$, i.e., the square below is commutative, for all RSG-morphism $f : G \longrightarrow G'$.





• If P is a profinite RSG and $h : G \longrightarrow P$ is a SG-morphism, there is a unique continuous SG-morphism, $\widetilde{h} : \mathcal{P}(G) \longrightarrow P$, such that the triangle above is commutative.

• The Profinite hull functor is left adjoint to the "forgetful" functor, $U : \mathbf{RSG}_{pf} \longrightarrow \mathbf{RSG}$, and the natural transformation $\eta : Id_{\mathbf{RSG}} \longrightarrow U \circ \mathcal{P}$ is the unit of this adjunction.

• Since the functor $U : \mathbf{RSG}_{pf} \longrightarrow \mathbf{RSG}$ is a right adjoint it preserves all limits. Hence, when we consider limits of profinite RSGs, we need not specify the limit as a profinite RSG (with the endowed topology) or as a (discrete) RSG. □

2 Local-global principles in the theory of special groups

Local-global principles in the algebraic theory of quadratic forms were developed, initially, in the context of fields. Of fundamental importance is Pfister's local-global principle for isometry in the reduced theory of quadratic forms, formulated and proven in the 60's (see [21]), a vast generalization of Sylvester's inertia law for forms with unit real coefficients. This was in turn generalized to reduced special groups in [9] and also to others *non first-order* codifications of the ATQF of fields as the theory of *abstract order spaces* introduced by M. Marshall (see [26]). In the 1980's, M. Marshall established some (strong) local-global principles such as the *Isotropy Theorem* and the *Extended*

Isotropy Theorem (see, for instance, Theorems 4.3.1 and 4.3.2 in [26]). Later, [27], formulated a broad local-global principle of *logical content*, called the "pp-conjecture" (Question 1 in [27]), that is considered in several works ([6], [17], [18], [19], [20], [27], [28]) and has a *negative* answer. It should be registered that "pp-conjecture" can only be formulated in the first-order language of special groups and not in the category of abstract order spaces.

The Boolean hull functor of reduced special groups, developed in [9], is an essential construction in the solution of a Marshall's conjecture exposed in [10] and it codifies Pfister's local-global principle. The profinite hull functor of special groups. introduced in [33], is a finer and more regular construction than the Boolean hull: it encodes a new (and stronger) local-global principle − the subform reflection property ([33]) − and preserves many properties and constructions of RSGs ([34], [35]).

We remark that all the former local-global principles (including the subform reflection property) can be described as restrictions of the general local global principle for pp-formulas. For the reader's convenience, we list the known local-global principles, in increasing order of strength.

15 Pfister's local-global principle ([9], [27])

• Usual description: $\forall G \in \mathbf{RSG}$ $\forall \varphi, \psi$ forms of the *same* dimension over G :

$$\varphi \equiv_G \psi \quad \Leftrightarrow \quad \text{for each } \sigma \in X_G = Hom_{SG}(G, \mathbb{Z}_2) \;\; \sigma\star\varphi \equiv_{\mathbb{Z}_2} \sigma\star\psi.$$

• Equivalent description: $\forall G \in \mathbf{RSG}$ $\forall \varphi, \psi$ forms of the *same* dimension over G :

$$\varphi \equiv_G \psi \quad \Leftrightarrow \quad \text{for each } \Sigma \in X(G) \;\; \varphi/\Sigma \equiv_{G/\Sigma} \psi/\Sigma.$$

• Functorial encoding by the Boolean Hull (B):

$\forall G \in \mathbf{RSG}$ the canonical SG-morphism $\varepsilon_G : G \longrightarrow B(G)$ is a *complete embedding.* □

16 Marshall's Isotropy Theorem ([25], [27]) For all $G \in$ RSG and all forms ϕ over G,

$$\phi \text{ is isotropic over } G \;\Leftrightarrow\; \forall\, \Delta \in \mathcal{F}(G),\ \phi/\Delta \text{ is isotropic over } G/\Delta.$$ □

17 Subform Reflection Property ([33])

• $\forall G \in \mathbf{RSG}$ $\forall \varphi, \psi$ forms of the *arbitrary* dimensions over G:

$$\varphi \preceq_G \psi \quad \Leftrightarrow \quad \text{for each } \Delta \in \mathcal{F}(G) \;\; \varphi/\Delta \preceq_{G/\Delta} \psi/\Delta.$$

• Functorial encoding by the Profinite Hull: $\forall G \in \mathbf{RSG}$, the canonical SG-morphism, $G \xrightarrow{\eta_G} \mathcal{P}(G)$, *reflects subforms.* □

18 The p.p. local-global principle

• Let $\mathbf{RSG}_{pp}$ be the subclass of $\mathbf{RSG}$ whose members are the RSGs, G, such that: for each pp-formula, $\phi(\vec{x})$, in L_{SG} and each $\vec{g} = (g_0, \ldots, g_{n-1}) \in G^n$: $G \models \phi[\vec{g}] \;\Leftrightarrow\; \forall\, \Delta \in \mathcal{F}(G),\;\; G/\Delta \models \phi[\vec{g}/\Delta]$.

• The *pp-conjecture*: $\mathbf{RSG}_{pp} = \mathbf{RSG}$ ([27]).

• $\mathbf{RSG}_{pp}$ is an elementary class of L_{SG}-structures ([6] and section 3 below).

• $\mathbf{RSG}_{pp}$ is *strictly* contained in $\mathbf{RSG}$ ([17], [18], [19]). □

3 On the subclass $\mathbf{RSG}_{pp} \subseteq \mathbf{RSG}$

In this last section we use the previously developed instruments on the profinite hull functor of RSGs, $\mathcal{P}$, to provide a new (and simpler) proof of a result established [6], employing deep model-theoretic tools: the class $\mathbf{RSG}_{pp}$ is an elementary class of L_{SG}-structures.

3.1 More on local-global principle for pp-formulas

We begin recalling some general results on L-pure embeddings appearing in [32].

Fact 3.1 *a) Let $M \xrightarrow{f} N \xrightarrow{g} P$ be L-morphisms. Then:*

(1) *f, g pure $\Rightarrow$ $g \circ f$ pure;*

(2) *$g \circ f$ pure $\Rightarrow$ f pure. In particular, every L-section in* **L-mod** *is a pure embedding.*

b) If $f_i : M_i \longrightarrow N_i$, $i \in I$, is a family of pure L-morphisms, their product, $\prod_{i\in I} f_i : \prod_{i\in I} M_i \longrightarrow \prod_{i\in I} N_i$, is a pure L-morphism.

c) Let $\langle I, \leq \rangle$ be an upward directed poset and $\mathcal{M} = \langle M_i; \{f_{ij} : i \leq j\} \rangle$ and $\mathcal{N} = \langle N_i; \{g_{ij} : i \leq j\} \rangle$ be I-diagrams in **L-mod**. *Let $\underrightarrow{lim}\ \mathcal{M} = \langle M; f_i \rangle$ and $\underrightarrow{lim}\ \mathcal{N} = \langle N; g_i \rangle$ their colimits in* **L-mod**. *Let $\langle h_i \rangle_{i\in I} : \mathcal{M} \longrightarrow \mathcal{N}$ be a morphism of I-diagrams and let $\underrightarrow{lim}\ h_i = h : M \longrightarrow N$ be the limit L-morphism. Then:*

(1) *If each h_i is pure, then $h : M \longrightarrow N$ is pure;*

(2) *If each f_{ij} is pure, $i \leq j$ in I, then $f_i : M_i \longrightarrow M$ is pure.*

d) Let $f : M \longrightarrow N$ be a L-morphism. Then the following conditions are equivalent:

(1) *f is a pure L-embedding;*

(2) *There are a L-structure P, a L-morphism $g : N \longrightarrow P$ and a* pure L-embedding *$h : M \longrightarrow P$ such that $g \circ f = h$;*

(3) *There are a L-structure P, a L-morphism $g : N \longrightarrow P$ and a* L-elementary embedding *$h : M \longrightarrow P$ such that $g \circ f = h$;*

(4) *There is an ultrafilter pair (I, U) and a L-morphism $g' : N \longrightarrow M^I/U$ such that $g' \circ f = \delta_M$, where $\delta_M : M \longrightarrow M^I/U$ is the canonical diagonal L-elementary embedding.*

□

The equivalent conditions below provide an useful criterion for a RSG G be in $\mathbf{RSG}_{pp}$.

Lemma 3.2 ([33]) *For a RSG, G, the following are equivalent:*

a) $\eta_G : G \longrightarrow \mathcal{P}(G)$ *is a pure* **SG**-*embedding.*
b) There are a profinite RSG, H*, and a pure* **SG**-*embedding,* $j : G \longrightarrow H$*.*
c) $\delta_G : G \longrightarrow \Pi(G)$ *is a pure* **SG**-*embedding.*
d) For each $\phi(v_1, \ldots, v_n)$ *in p.p.*(L_{SG}) *and* $\overline{g} \in G^n$*,*

$$G \models \phi[\overline{g}] \quad \Leftrightarrow \quad \text{for all } \Delta \in \mathcal{F}(G),\ G/\Delta \models \phi[p_\Delta(\overline{g})].$$

e) For each $\phi(v_1, \ldots, v_n)$ *in p.p.*(L_{SG}) *and* $\overline{g} \in G^n$*,*

$$G \not\models \phi[\overline{g}] \quad \Rightarrow \quad \exists\ \Delta \in \mathcal{F}(G), \text{ such that } G/\Delta \not\models \phi[p_\Delta(\overline{g})] \qquad .$$

Proof. Note that:
(†) The satisfaction of pp-formulas with parameters is preserved by L_{SG}-morphisms.
(1) We first prove the equivalence between (a), (b) and (c).
Note that (a) ⇒ (b) is clear, while (c) ⇒ (b) follows directly from (†) and the equation $\delta_G = \iota \circ \eta_G$.
(b) ⇒ (a) : Given j as in (b), by 14, there is $\widetilde{j} : \mathcal{P}(G) \longrightarrow H$, such that $\widetilde{j} \circ \eta_G = j$. Thus, if j is a pure **SG**-embedding, the same must be true of η_G.
(a) ⇒ (c): Since $\mathcal{P}(G) = \varprojlim_{\Delta \in \mathcal{F}(G)} G/\Delta$, $\Pi(G) = \prod_{\Delta \in \mathcal{F}(G)} G/\Delta$,

$$\iota : \varprojlim_{\Delta \in \mathcal{F}(G)} G/\Delta \longrightarrow \prod\nolimits_{\Delta \in \mathcal{F}(G)} G/\Delta$$

has a retract (11) and $\delta_G = \iota \circ \eta_G$, if η_G is a pure **SG**-embedding, the same must be true of δ_G.
(2) The equivalence between (c), (d) and (e) follows immediately from the definition of satisfaction on product structures and (†) above. ■

Before going on with the technical results and in the hope of shedding some light into the contents of this section, we register the remarks that follow.

Remark 3.3 *a) As seen above, the class* $\mathbf{RSG}_{pp}$ *of reduced special groups satisfying the local-global for pp-formulas coincides with the class of RSGs such that the canonical arrow into its profinite hull is a pure embedding. This kind of result can be stated and proved in a vastly general context: a certain kind of elementary class* $\mathcal{A}$ *of L-structures, for an arbitrary first-order language (see [32]). As profinite structures are pure injective structures (Theorem 8 in [32]), it is natural consider the subclass* $\mathcal{A}_{pp} \subseteq \mathcal{A}$ *of the (discrete) structures,* $M \in \mathcal{A}$*, such that the canonical arrow,* $\eta_M : M \longrightarrow \mathcal{P}(M)$*, is a* L-pure embedding*. This can be rephrased as a "local-global principle": for each formula* $\phi(\vec{x}) \in p.p.(L)$ *and any finite sequence* $\vec{a}$ *in* M *the following are equivalent, for each "saturated" congruence such that* $M/C \in \mathcal{A}_{fin}$*:*

$$M \models \phi[\vec{a}] \quad \Leftrightarrow \quad M/C \models \phi[\vec{a}/C].$$

b) Every Boolean algebra satisfies the pp-local-global principle (i.e. $\mathbf{BA}_{pp} = \mathbf{BA}$*). In [33] it is shown that the BA-homomorphism* $\eta_B : B \longrightarrow \mathcal{P}(B)$ *can be identified with the*

injective BA-homomorphism $B \rightarrowtail 2^{Stone(B)}$, $b \mapsto \{U \in Stone(B) : b \in U\}$, *which gives the usual Stone topology on the set* $Stone(B) = \{U \subseteq 2^B : U$ *is an ultrafilter in* $B\}$. *But, by Corollary 2.2 (p. 951) in [12], any BA-monomorphism is a* L_{BA}*-pure embedding. In general, the canonical arrow,* $\eta_B : B \longrightarrow \mathcal{P}(B)$, does not *yield the (pure) injective hull of B: just consider a (complete =) injective BA, B, that is not isomorphic any power set algebra* 2^X.
c) Concerning structures M in the subclass $\mathcal{A}_{pp} \subseteq \mathcal{A}$, *the map* $\eta_M : M \longrightarrow \mathcal{P}(M)$ *is the canonical L-pure embedding into an L-injective pure structure but, in general,* η_M is not *the pure injective hull of M: if the epimorphisms in the category* $\mathcal{A}_{pp}$ *are surjective maps and not all pure injective structures in* $\mathcal{A}_{pp}$ *are profinite, then this follows from Theorem 3.2 in [1] (e.g., in the category* $\mathbf{BA}_{pp} = \mathbf{BA}$ *both of the above conditions are satisfied).* □

In the what follows, we apply the above results to show that, although $\mathbf{RSG}_{pp} \subsetneq \mathbf{RSG}$, $\mathbf{RSG}_{pp}$ contains some important classes of RSGs.

Proposition 3.4 *If G is a RSG associated to a Boolean algebra (see 7), then* $G \in \mathbf{RSG}_{pp}$.

Proof. We have already remarked, in 8, that a map $f : A \to B$ between RSGs associated to Boolean algebras is a BA-morphism iff is it is a RSG-morphism. An analogous statement holds for pure embeddings instead homomorphisms: indeed, by 8, the class of RSGs associated to a BA is a L_{SG}-elementary class, thus it is closed under ultraproducts and then the equivalence f is a L_{BA}-pure embedding $\Leftrightarrow$ f is a L_{SG}-pure embedding, follows from Fact 3.1.(d). By 3.3.(b), $\mathbf{BA}_{pp} = \mathbf{BA}$. Putting this two results together, we obtain $\mathbf{BA}$ "$\subseteq$" $\mathbf{RSG}_{pp}$. ■

Proposition 3.5 *If G is a RSG with finite chain length, then* $G \in \mathbf{RSG}_{pp}$.

Proof. If G is a RSG with finite chain length, then it satisfies the hypothesis of Proposition 3.1.11 in [2], whence the mapping

$$\nu : G \longrightarrow \prod_{p \in X_G^+} G/\ker(p), \text{ given by } \nu(g) = \langle g/ker(p) \rangle_{p \in X_G^+}$$

is a a L_{SG}-pure embedding, where $X_G^+ := \bigcup\{X_G^H : H$ is a finite $\mathbf{SG}$-retract of $G\}$, and for $H \subseteq G$, $X_G^H := \{G \overset{p}{\to} \longrightarrow H : p$ is a $\mathbf{SG}$-morphism and $p \circ \iota_H = Id_H\}$.

Now, by Lemma 3.2, it is enough to show that $\prod_{p \in X_G^+} G/\ker(p)$ is a *profinite* RSG. If H is a finite $\mathbf{SG}$-retract of G, p is a $\mathbf{SG}$-morphism and $p \circ \iota_H = Id_H$, then H is a RSG and p is a surjective *regular* $\mathbf{SG}$-morphism (Definition 2.22 in [9]) and, by Proposition 2.23 in [9], $\bar{p} : G/\ker(p) \to H$ is an isomorphism of *finite reduced* special groups. Since G is reduced, $X_G \neq$ and, as $X_G = X_G^{\mathbb{Z}_2} \subseteq X_G^+$, we conclude X_G^+ to be a non-empty set. Since the class of profinite RSGs contains the finite RSGs and is closed under non-empty products, it follows that $\prod_{p \in X_G^+} G/\ker(p)$ is a *profinite* RSG, as needed. ■

Remark 3.6 *a) By Proposition 2.3 in [27], if a RSG is in* $\mathbf{RSG}_{pp}$*, then all its* group extensions *are in* $\mathbf{RSG}_{pp}$*; other closure properties of* $\mathbf{RSG}_{pp}$ *are also established therein.*
b) The class $\mathbf{RSG}_{pp}$ *contains the class of RSGs associated to BAs, which coincides with the class of RSGs with* stability index $\leqslant 1$. *On the other hand, the class RSGs with* finite *stability index satisfies the restriction of the pp-conjecture with respect to set of pp-formulas named* product free and 1-related *([28]).* □

3.2 The class $\mathbf{RSG}_{pp}$ is elementary

Combining results on the profinite hull functor of RSGs with the equivalences in Lemma 3.2, we obtain a new proof that $\mathbf{RSG}_{pp}$ is an elementary class of L_{SG}-structures. We begin by analyzing the behavior of $\mathbf{RSG}_{pp}$ under directed colimits. Since **RSG** is axiomatizable by geometric sentences, **RSG** is closed under colimits of arbitrary upward directed systems. For instance:

Example 3.7 *Let* G *be a RSG and* $\mathcal{S} \subseteq Ssat^\star(G)$ *an upward directed set of proper saturated subgroups of* G*. Then* $\Theta := \bigcup\{\Sigma : \Sigma \in \mathcal{S}\}$ *is a proper saturated subgroup and* G/Θ *is an RSG, canonically isomorphic to the inductive limit of the directed system* $\ll \ (G/\Sigma \overset{p_{\Sigma\Sigma'}}{\twoheadrightarrow} G/\Sigma') : (\Sigma \subseteq \Sigma') \in \mathcal{S} \ \rangle$. □

For the proof of Proposition 3.10, we shall need some technical results on directed inductive limits. Recall that if (Y, H) is an AOS, the subspace (Y', H') generated by a subset S of Y is given by $Y' = \bigcap\{[a = 1] : a \in ker(\sigma), \forall \sigma \in S\}$, where $[a = 1] := \{\tau \in Y : \tau(a) = 1\}$.

Lemma 3.8 *If we stratify the set of pp-formulas in* L_{SG} *by the number of quantified variables (q) and the number of free variables or parameters (p) then there is a* uniform *bound* $B'(q, p)$ *such that for each pp-formula* $\phi(\vec{x}) = \exists \vec{y}(atom_1(\vec{x}, \vec{y}) \wedge \ldots \wedge atom_k(\vec{x}, \vec{y}))$*, with* $length(\vec{y}) = l \leq q$ *and* $length(\vec{x}) = m \leq p$*, each space of orderings* (X, G) *(i.e., a RSG* G*) and each* $\vec{g}$ *in* G^m*:*

(†) *If there is a* finite *subspace* (X', G') *of* (X, G) *(i.e.,* $G' \cong G/\Delta$ *for some* $\Delta \in Ssat^\star(G)$*) such that* $G/\Delta \nvDash \phi[\vec{g}/\Delta]$*, then there is a* finite *subspace* (X'', G'') *of* (X, G) *(*$G'' \cong G/\Gamma$ *for some* $\Gamma \in Ssat^\star(G)$*), such that* $G/\Gamma \nvDash \phi[\vec{g}/\Gamma]$ *and the minimum number of* generators *of* (X'', G'') *is* $\leq B'(q, p)$.

Proof. Lemma 4 in [6] yields a similar finite bound, $B(q, p)$, for the cardinality of the finite subspaces satisfying $\neg\phi[\cdot]$. Since for an abstract order space (Y, H), we have $card(Y) \leq 2^{card(H)}$ and $card(H) \leq 2^{card(Y)}$ (whence Y is finite iff H is finite), there is a *finite* number of isomorphism classes of finite AOSs (i.e., isomorphism classes of finite RSGs), whose cardinality are all bounded by $B(q, p)$. Thus, the sup of minimum number of generators of these finite spaces is attained; this is precisely $B'(q, p)$. ■

Fact 3.9 *Let* $\langle I, \leq \rangle$ *be an* upward *directed poset, let* $\mathcal{G} : \langle I, \leq \rangle \longrightarrow \mathbf{RSG}$ *be a diagram* $\mathcal{G} = \ll (G_i \overset{f_{ij}}{\to} \longrightarrow G_j) : (i \leq j) \in I \rangle$ *and let* $\{(G_i \overset{f_i}{\to} \longrightarrow G) : i \in I\} = \varinjlim \mathcal{G}$ *be the inductive limit in* **RSG** *(notation* $G := \varinjlim G_i$*). Note that* $\ll (Ssat(G_j) \overset{f^*_{ij}}{\to} \longrightarrow Ssat(G_i)) : (i \leq j) \in I \rangle$ *and* $\ll (X_{G_j} \overset{f^\star_{ij}}{\to} \longrightarrow X_{G_i}) : (i \leq j) \in I \rangle$ *are both* downward *directed systems.*

Consider the posets (by pointwise inclusion):

- $Ssat(\mathcal{G}) := \varprojlim Ssat(G_i) = \{(\Delta_i)_{i \in I} \in \prod_{i \in I} Ssat(G_i) : \forall i \leq j \in I, f^*_{ij}(\Delta_j) = \Delta_i\}$;
- $Ssat^\star(\mathcal{G}) := \varprojlim Ssat^\star(G_i) = \varprojlim Ssat(G_i) \cap \prod_{i \in I} Ssat^\star(G_i)$;
- *For each* $n \in \mathbb{N}$, $\mathcal{F}_{\leq n}(\mathcal{G}) := \varprojlim \mathcal{F}_{\leq n}(G_i) = \varprojlim Ssat(G_i) \cap \prod_{i \in I} \mathcal{F}_{\leq n}(G_i)$;
- $X(\mathcal{G}) := \varprojlim X(G_i) = \varprojlim Ssat(G_i) \cap \prod_{i \in I} X(G_i)$; • $\mathcal{F}(\mathcal{G}) := \bigcup_{n \in \mathbb{N}} \mathcal{F}_{\leq n}(\mathcal{G})$;
- $X_{\mathcal{G}} := \varprojlim X_{G_i} = \{(\sigma_i)_{i \in I} \in \prod_{i \in I} X_{G_i} : \forall i \leq j \in I, f^\star_{ij}(\sigma_j) \ (= \sigma_j \circ f_{ij}) = \sigma_i\}$.

Consider the (increasing) mappings : $Ssat(G) \rightleftarrows Ssat(\mathcal{G})$
$D : Ssat(\varinjlim G_i) \longrightarrow \varprojlim Ssat(G_i)$*, given by* $\Delta \mapsto (f^*_i(\Delta))_{i \in I}$ *and*
$C : \varprojlim Ssat(G_i) \longrightarrow Ssat(\varinjlim G_i)$*, given by* $(\Delta_i)_{i \in I} \mapsto \varinjlim \Delta_i = \bigcup_{i \in I} f_i[\Delta_i]$.

Then:
a) D and C are well defined; if $(\Delta_i)_{i \in I} \in Ssat(\mathcal{G})$*, then* $(\varinjlim G_i)/(\varinjlim \Delta_i) \cong \varinjlim (G_i/\Delta_i)$.
b) If $\Delta \in Ssat(G)$*, then* $\Delta = \varinjlim f^*_i(\Delta) = \bigcup_{i \in I} f_i[f^*_i(\Delta)]$ *(i.e.* $C \circ D = id$*). If* $(\Delta_i)_{i \in I} \in Ssat(\mathcal{G})$*, then* $\forall j \in I$, $\Delta_j \subseteq f^*_j(\varinjlim \Delta_i)$.
c) The maps C and D yield, by restriction, the following maps:
$Ssat^\star(G) \rightleftarrows Ssat^\star(\mathcal{G})$, $\mathcal{F}_{\leq n}(G) \rightleftarrows \mathcal{F}_{\leq n}(\mathcal{G})$, $\mathcal{F}(G) \rightleftarrows \mathcal{F}(\mathcal{G})$, $X(\mathcal{G}) \rightleftarrows X(G)$.
d) The restrictions $X(\mathcal{G}) \rightleftarrows X(G)$ *are inverse* bijections *and yield* homeomorphisms *of Boolean spaces,* $X_{\mathcal{G}} \rightleftarrows X_G$. □

Proposition 3.10 $\mathbf{RSG}_{pp}$ *is closed by (upward) directed inductive limits.*

Proof. Instead of the deep model-theoretic tools used in the proof of Proposition 6.1 in [6], we provide a "topological" flavored one, generalizing that in Theorem 2.2.1 in [17].

We must show that for each *upward* directed poset $\langle I, \leq \rangle$ and each diagram $\mathcal{G} := \ll (G_i \overset{f_{ij}}{\to} \longrightarrow G_j) : (i \leq j) \in I \rangle$ such that $G_i \in \mathbf{RSG}_{pp}$, $\forall i \in I$, then $G := \varinjlim G_i \in \mathbf{RSG}_{pp}$. Assume, *to get a contradiction*, that there is upward directed system $\mathcal{G}$ whose components are in $\mathbf{RSG}_{pp}$, a p.p. formula $\phi(\vec{x})$ and parameters $\vec{g}$ in $G := \varinjlim G_i$, such that $G \nvDash \phi[\vec{g}]$, but for all $\Delta \in \mathcal{F}(G)$, $G/\Delta \vDash \phi[\vec{g}/\Delta]$. We split the proof into several steps.

(1) Write $b := B'(q,p)$ and let $C := \{$equivalence classes modulo isomorphism of the finite RSGs with minimum number of generator $\leq b$ $\}$; we saw above that C is finite. Now we select a (finite) subset $C_G \subseteq Ssat(G)$, such that if $\Delta \in C_G$, then the class of G/Δ is in C and distinct elements of C_G are associated to distinct classes in C. Since for each $\Delta \in C_G$, G/Δ is finite, we may choose a finite subset of representatives, $\vec{h}_\Delta$ in G^p, so that $G/\Delta \vDash \exists\vec{y}(atom_1(\vec{x},\vec{y}) \wedge \ldots \wedge atom_k(\vec{x},\vec{y}))[\vec{x}|\vec{h}_\Delta/\Delta]$ (i.e., $G/\Delta \vDash \phi[\vec{h}_\Delta/\Delta]$) and then a finite set of representatives, $\vec{t}_\Delta$ in G^q, such that $G/\Delta \vDash atom_1(\vec{t}_\Delta/\Delta, \vec{h}_\Delta/\Delta) \wedge \ldots \wedge atom_k(\vec{t}_\Delta/\Delta, \vec{h}_\Delta/\Delta)$.

Since $(\varinjlim G_i)/\Delta \cong \varinjlim_{i\in I} (G_i/f_i^*(\Delta))$ (Fact 3.9.(a) above), the set of atomic formulas in ϕ, as well as the two set of representatives fixed above are all finite and $\langle I, \leq \rangle$ is upward directed, there are $j \in I$ and a two finite sets of parameters, $\vec{h}_{\Delta_j} \in G_j^p$ and $\vec{t}_{\Delta_j} \in G_j^q$, that are "liftings" to G_j^p of the parameters $\vec{h}_\Delta$, $\vec{t}_\Delta$ and also satisfying $G_j/\Delta_j \vDash atom_1(\vec{t}_{\Delta_j}/\Delta_j, \vec{h}_{\Delta_j}/\Delta_j) \wedge \ldots \wedge atom_k(\vec{t}_{\Delta_j}/\Delta_j, \vec{h}_{\Delta_j}/\Delta_j)$, where $\Delta_j := f_j^*(\Delta) \in Ssat(G_j)$. Now, because C_G is finite and $\langle I, \leq \rangle$ is directed, we can choose i_0 above all the j associated to $\Delta \in C_G$ and then two finite sets of parameters $\vec{h}_{\Delta_{i_0}}$, $\vec{t}_{\Delta_{i_0}} \in G_{i_0}^q$ that are "liftings" of the parameters $\vec{h}_\Delta$ and $\vec{t}_\Delta$ to G_{i_0} and such that $G_{i_0}/\Delta_{i_0} \vDash atom_1(\vec{t}_{\Delta_{i_0}}/\Delta_{i_0}, \vec{h}_{\Delta_{i_0}}/\Delta_{i_0}) \wedge \ldots \wedge atom_k(\vec{t}_{\Delta_{i_0}}/\Delta_{i_0}, \vec{h}_{\Delta_{i_0}}/\Delta_{i_0})$, where $\Delta_{i_0} := f_{i_0}^\star(\Delta) \in Ssat(G_{i_0})$.

(2) Since $G \nvDash \phi[\vec{g}]$, for each $j \geq i_0$ and each representative $\vec{g}_j$ in G_j^p of $\vec{g}$ in G (i.e., $\vec{g} = f_j(\vec{g}_j)$), we have $G_j \nvDash \phi[\vec{g}_j]$ (because ϕ is pp-formula and f_j is a L_{SG}-homomorphism). By hypothesis, we also have $G_j \in \mathbf{RSG}_{pp}$, and so Lemma 3.8 yields a finite subspace of (X_j, G_j), with a generating set S_j so that $card(S_j) \leq b$ and $G_j/\Sigma_j \nvDash \phi[\vec{g}_j/\Delta_j]$ (where $\Sigma_j = S_j^\perp := \bigcap\{ker(\sigma) : \sigma \in S_j\}$). For each $j \geq i_0$, we choose S_j as above and fix a surjective map $\{1,\ldots,b\} \twoheadrightarrow S_j$. Write S_j for $\{\sigma_j^1,\ldots,\sigma_j^b\}$.

(3) For each $j \geq i_0$, consider the AOS-morphism $f_{i_0,j}^\star : X_j \longrightarrow X_{i_0}$. Since the set $\langle I, \leq \rangle$ is upward directed, the subset $I_{i_0} := \{j \in I : j \geq i_0\}$ is cofinal in I and hence also upward directed. Thus, for each $a \leqslant b$, the map $j \in I_{i_0} \overset{s_a}{\to}\mapsto f_{i_0,j}^\star(\sigma_j^a)$ is a *net* in the *Boolean* space X_{i_0}. Since any net in a compact space has a convergent subnet and the set $\{s_a : a \leq b\}$ is finite, there is a downward directed set I' and a increasing cofinal map, $I' \longrightarrow I_{i_0}$, that is the common domain of subnets $s'_a : I' \longrightarrow X_{i_0}$ of the nets s_a, with $a \leq b$, and such that the s'_a converge to some $\sigma^a \in X_{i_0}$, for each $a \leq b$.

(4) A compactness argument (below), will yield a "lifting" $\tau^a \in X_G$ of $\sigma^a \in X_{i_0}$ (i.e. $f_{i_0}^\star(\tau^a) = \sigma^a$), for each $a \leqslant b$. To this end, we must prove there is a non-empty fiber of $\sigma^a \in X_{i_0}$ under $f_{i_0}^\star : X_G \longrightarrow X_{i_0}$. By Fact 3.9.(e) and the cofinality in I of the subset $I_{i_0} := \{j \in I : j \geq i_0\}$, this is equivalent to showing the non-emptyness of the fiber of

$\sigma^a \in X_{i_0}$ under the projection $p_{i_0} : \varprojlim_{j \in I_{i_0}} X_j \longrightarrow X_{i_0}$.

For each $k, j \in I_{i_0}$ such that $k \geq j$, consider $T_{jk} := \{(\sigma_i)_{i \geq i_0} \in \prod_{i \geq i_0} X_i : f^\star_{jk}(\sigma_k)\ (= \sigma_k \circ f_{jk}) = \sigma_j\}$. Then T_{jk} is a closed subset of the Boolean product space and, because $\varprojlim X_j = \bigcap_{k \geq j \geq i_0} T_{jk}$, the fiber $p^{-1}_{i_0}[\{\sigma^a\}]$ is a closed subset of the compact space $\prod_{i \geq i_0} X_i$; hence, it is empty iff there is a finite set $k_1 \geq j_1, \ldots, k_n \geq j_n$ in I_{i_0} such that $= \pi^{-1}_{i_0}[\{\sigma^a\}] \cap T_{j_1k_1} \cap \ldots \cap T_{j_nk_n}$. Now assume $p^{-1}_{i_0}[\{\sigma^a\}] =$; since I is directed, if $l \geqslant k_1, \ldots k_n$, then, σ^a *is not a member of the closed set* $\pi_{i_0}[T_{j_1k_1} \cap \ldots \cap T_{j_nk_n} \cap T_{k_1l} \cap \ldots \cap T_{k_nl}]$. By the regularity of Boolean spaces, there is an open neighborhood V of σ^a such that $= V \cap \pi_{i_0}[T_{j_1k_1} \cap \ldots \cap T_{j_nk_n} \cap T_{k_1l} \cap \ldots \cap T_{k_nl}]$. Then, as there is a increasing cofinal map $I' \longrightarrow I_{i_0}$ that is the common domain of (sub)nets $s'_a : I' \longrightarrow X_{i_0}$ such that the s'_a is converging to $\sigma^a \in X_{i_0}$, there is $l' \in I_{i_0}$, $l' \geq l$ and $\sigma^a_{l'} \in X_{l'}$ such that $f^\star_{i_0l'}(\sigma^a_{l'}) \in V$. Hence, if we consider any $\vec{s} \in \prod_{i \geq i_0} X_i$ such that for i with $i_0 \leq i \leq l'$, $s_i := f^\star_{il'}(\sigma^a_{l'})$, we obtain $\vec{s} \in \pi^{-1}_{i_0}[V] \cap T_{j_1k_1} \cap \ldots \cap T_{j_nk_n} \cap T_{k_1l} \cap \ldots \cap T_{k_nl}$, a contradiction, that establishes (4).

(5) Let $S := \{\tau^a : a \leq b\} \subseteq X_G$ and let $\Gamma := S^\perp = \bigcap\{ker(\tau^a) : a \leq b\} \in Ssat(G)$; then, $(X_{G/\Gamma}, G/\Gamma)$ is the subspace of (X_G, G) generated by S. Since $card(S) \leq b$, by item (1) above, there is $\Delta \in C_G$ with $G/\Delta \cong G/\Gamma$ and so $(X_{G/\Delta}, G/\Delta)$ is also a subspace of (X_G, G) with at most b generators, let's say, by $S' := \{\tau'^a : a \leq b\}$. By the hypothesis on G, we have $G/\Delta \vDash \phi[\vec{g}/\Delta]$ and in part (1) of the proof we have selected a finite subset of G^p and a finite subset of G^q of representatives, $\vec{g}_\Delta$ in G^p, $\vec{t}_\Delta$ in G^q so that $G/\Delta \vDash atom_1(\vec{t}_\Delta/\Delta, \vec{g}_\Delta/\Delta) \wedge \ldots \wedge atom_k(\vec{t}_\Delta/\Delta, \vec{g}_\Delta/\Delta)$. By the choice of i_0 in (1), there are two finite sets of parameters $\vec{g}_{\Delta_{i_0}} \in G^p_{i_0}$ and $\vec{t}_{\Delta_{i_0}} \in G^q_{i_0}$ that are "liftings" of the parameters $\vec{g}_\Delta$ and $\vec{t}_\Delta$ to G_{i_0} and such that $G_{i_0}/\Delta_{i_0} \vDash atom_1(\vec{t}_{\Delta_{i_0}}/\Delta_{i_0}, \vec{g}_{\Delta_{i_0}}/\Delta_{i_0}) \wedge \ldots \wedge atom_k(\vec{t}_{\Delta_{i_0}}/\Delta_{i_0}, \vec{g}_{\Delta_{i_0}}/\Delta_{i_0})$, where $\Delta_{i_0} := f^*_{i_0}(\Delta) \in Ssat(G_{i_0})$. Moreover, if $\Delta^\perp_{i_0} := \{\sigma \in X_{i_0} : \Delta_{i_0} \subseteq ker(\sigma)\}$, then $\{\sigma^a : a \leq b\} \subseteq \Delta^\perp_{i_0}$.

(6) Since for RSGs, the atomic formulas $atom(\vec{x}, \vec{y})$ are equivalent to formulas $u(\vec{x}, \vec{y}) \in D\langle 1, v(\vec{x}, \vec{y}) \rangle$, where u, v are L_{SG}-terms (see subsection 1.1), we can write

$$G_{i_0}/\Delta_{i_0} \vDash \bigwedge_{l \leq k} u_l(\vec{t}_{\Delta_{i_0}}/\Delta_{i_0}, \vec{g}_{\Delta_{i_0}}/\Delta_{i_0}) \in D\langle 1, v_l(\vec{t}_{\Delta_{i_0}}/\Delta_{i_0}, \vec{g}_{\Delta_{i_0}}/\Delta_{i_0}) \rangle$$

which, by Pfister's local-global principle, is equivalent to the inclusion $\Delta^\perp_{i_0} \subseteq A_{i_0}$ where $A_{i_0} := \bigcap_{l \leq k}[u_l(\vec{t}_{\Delta_{i_0}}, \vec{g}_{\Delta_{i_0}}) = 1] \cup [-v_l(\vec{t}_{\Delta_{i_0}}, \vec{g}_{\Delta_{i_0}}) = 1]$. Note that A_{i_0} is a clopen in X_{i_0}; since $\{\sigma^a : a \leq b\} \subseteq A_{i_0}$ (by (5)), item (3) yields $j \geq i_0$ so that $\{f^\star_{i_0,j}(\sigma^a_j) : a \leq b\} \subseteq A_{i_0}$. As $f^\star_{i_0,j} : X_j \longrightarrow X_{i_0}$ satisfies $(f^\star_{i_0,j})^{-1}[A_{i_0}] = A_j$, where $A_j := \bigcap_{l \leq k}[u_l(f_{i_0,j}\vec{t}_{\Delta_{i_0}}, f_{i_0,j}\vec{g}_{\Delta_{i_0}}) = 1] \cup [-v_l(f_{i_0,j}\vec{t}_{\Delta_{i_0}}, f_{i_0,j}\vec{g}_{\Delta_{i_0}}) = 1]$, we obtain $S_j = \{\sigma^a_j : a \leq b\} \subseteq A_j$. As in (2), $\Gamma_j = (S_j)^\perp$ and Pfister's local-global principle entails $G_j/\Gamma_j \vDash \bigwedge_{l \leq k} u_l(f_{i_0,j}\vec{t}_{\Delta_{i_0}}/\Gamma_j, f_{i_0,j}\vec{g}_{\Delta_{i_0}}/\Gamma_j) \in D\langle 1, v_l(f_{i_0,j}\vec{t}_{\Delta_{i_0}}/\Gamma_j, f_{i_0,j}\vec{g}_{\Delta_{i_0}}/\Gamma_j) \rangle$, i.e., with $\vec{g}_j := f_{i_0,j}\vec{g}_{\Delta_{i_0}}$, $G_j/\Gamma_j \vDash \phi[\vec{g}_j/\Gamma_j]$, contradicting (2) and ending the proof. ■

The proofs of the following results on quotients of RSGs will appear in [35].

Fact 3.11 *a) Let $f : G \to G'$ be a RSG-morphism, $\Delta \in Ssat(G)$ and consider $\Delta' := f_*(\Delta)$ = the saturated subgroup of G' generated by Δ. If f is a pure embedding, then $\Delta = f^{-1}[\Delta']$ and the (well-defined) map $f_{\Delta,\Delta'} : G/\Delta \to G'/\Delta'$, $g/\Delta \mapsto f(g)/\Delta'$ is a pure embedding.*
b) If H is a profinite RSG and $\Sigma \subseteq H$ is a saturated subgroup that satisfies

$$\textbf{(TSP)} \quad \Sigma = \bigcap\{\Delta \in \mathcal{V}(H) : \Sigma \subseteq \Delta\},$$

then the topological L_{SG}-structure quotient H/Σ is a profinite RSG. For instance, if P is a Pfister form over H, then $D_H(P) \subseteq H$ is a saturated subgroup that satisfies **(TSP)** *and thus $H/D_H(P)$ is a profinite RSG.*
c) If $G \in$ **RSG***, then canonical* **RSG***-morphism $\eta_{G/\Theta} : G/\Theta \longrightarrow \mathcal{P}(G/\Theta)$, has kernel $\Sigma_\Theta :=$*
$\bigcap\{ker(\mathcal{P}(G) \overset{p_\Delta}{\to} \longrightarrow G/\Delta) : \Theta \subseteq \Delta \in \mathcal{F}(G)\}$ and Σ_Θ is the least saturated subgroup above $(\eta_G)_\star(\Theta)$ that satisfies **(TSP)***. Moreover, $\eta_{G/\Theta} : G/\Theta \longrightarrow \mathcal{P}(G/\Theta)$ it is naturally isomorphic to the "derived"* **RSG***-morphism $(\eta_G)_{\Theta,\Sigma_\Theta} : G/\Theta \longrightarrow \mathcal{P}(G)/\Sigma_\Theta$, $g/\Theta \mapsto \eta_G(g)/\Sigma_\Theta$.* □

Lemma 3.12 *Let $G \in$* **RSG***, let P be a Pfister's form over G and write $\Theta := D_G(P)$. Then:*
a) $(\eta_G)_\star(\Theta)$ is a saturated subgroup of $\mathcal{P}(G)$ satisfying **(TSP)***.*
b) If η_G is pure SG-embedding, the same is true of $\eta_{G/\Theta}$.

Proof. a) Since P is a Pfister form over G and G is reduced , we have $D_G(P) \in Ssat(G)$ and it follows immediately from the definition of *direct image* that $(\eta_G)_\star(D_G(P)) = D_{\mathcal{P}(G)}(\eta_G\star P)$; since $\eta_G\star P$ is Pfister form over the profinite RSG $\mathcal{P}(G)$, Fact 3.11.(b) entails $D_{\mathcal{P}(G)}(\eta_G\star P)$ satisfies **(TSP)**. For (b), note that (a) and Fact 3.11.(c) yield $\Sigma_\Theta = (\eta_G)_\star(\Theta)$; the conclusion now follows from items (a) and (b) in Fact 3.11. ■

Proposition 3.13 **RSG**$_{pp}$ *is closed under quotients by proper saturated subgroups.*

Proof. We must prove that for each $G \in RSG$ and $\Theta \in Ssat^\star(G)$, if η_G is a pure SG-embedding, then the same is true of $\eta_{G/\Theta}$. By Lemma 3.12 above, this is true if $\Theta = D_G(P)$ for some P a (non-isotropic) Pfister's form over G. The general case can be obtained from this special case and Proposition 3.10: indeed, by Proposition 2.17 in [9], $\Theta = \bigcup\{D_G(P) : P \in Pfister(\Theta)\}$ is an upward directed union and so by 3.7, G/Θ is canonically isomorphic to the inductive limit of the directed system $\ll (G/D_G(P) \overset{p_{PP'}}{\to}\twoheadrightarrow G/D_G(P')) : P, P' \in Pfister(\Theta)$ and $D_G(P) \subseteq D_G(P') \rangle$. ■

Our main results are contained in the following Theorems:

Theorem 3.14 *The category $\mathbf{RSG}_{pp}$ has the following properties:*

a) $\mathbf{RSG}_{pp}$ is closed under isomorphisms. b) $\mathbf{RSG}_{pp}$ is closed under $\mathbf{SG}$-pure subgroups.
c) $\mathbf{RSG}_{pp}$ is closed under non-empty products. d) $\mathbf{RSG}_{pp}$ is closed under direct inductive limits.
e) $\mathbf{RSG}_{pp}$ is closed under quotients by proper saturated subgroups.
f) $\mathbf{RSG}_{pp}$ is closed under reduced products. g) $\mathbf{RSG}_{pp}$ is closed under elementary equivalence.

Proof. Items (a) and (b) are equivalent to the statement: for each $G \in \mathbf{RSG}_{pp}$, if G' is a SG and there is a $\mathbf{SG}$-pure embedding, $j : G' \longrightarrow G$, then $G' \in \mathbf{RSG}_{pp}$; this follows from Lemma 3.2 , because the composition of pure embeddings is a pure embedding (Fact 3.1.(a)).
c) Let I be a non-empty set and $\{G_i : i \in I\} \subseteq \mathbf{RSG}_{pp}$; since $\mathbf{RSG}_{pf}$ is closed under non-empty products, $\prod_{i\in I} \mathcal{P}(G_i)$ is a profinite RSG and since the product of pure embeddings is a pure embedding (3.1.(b)), $\prod_{i\in I} \eta_{G_i} : \prod_{i\in I} G_i \longrightarrow \prod_{i\in I} \mathcal{P}(G_i)$ is a pure embedding. Now, Lemma 3.2 entails $\prod_{i\in I} G_i \in \mathbf{RSG}_{pp}$.
Items (d) and (e) were proven in Propositions 3.10 and 3.13, respectively.
f) Let $\mathcal{F}$ be a proper filter over a set $I \neq$ and $\{G_i : i \in I\} \subseteq \mathbf{RSG}_{pp}$; it is well-known that the reduced product $(\prod_{i\in I} G_i)/\mathcal{F}$ is isomorphic to the inductive limit of the direct system $\ll (\prod_{i\in J} G_i \overset{proj_{JK}}{\twoheadrightarrow} \prod_{i\in K} G_i) : (J \supseteq K) \in \mathcal{F} \rangle$ (see, for instance [31]), and the conclusion follows from items (a), (c) and (d). Alternatively, the canonical projection $p : \prod_{i\in I} G_i \twoheadrightarrow (\prod_{i\in I} G_i)/\mathcal{F}$ is a surjective regular $\mathbf{SG}$-morphism whose kernel is a proper saturated subgroup $\Delta_{\mathcal{F}}$, and so by Proposition 2.23 in [9], $(\prod_{i\in I} G_i)/\mathcal{F} \cong (\prod_{i\in I} G_i)/\Delta_{\mathcal{F}}$ and result is a consequence of items (a), (c) and (e).
g) By Fraine's Lemma (Lemma 8.1.1 in [7]), $G \equiv H$ iff G is elementary embeddable in some ultrapower of H and the conclusion follows from (a), (b) and (f). ■

Theorem 3.15 $\mathbf{RSG}_{pp}$ *is an elementary class in the language L_{SG}. Moreover, it can be axiomatizable by sets of: a) Horn-sentences* [3] *or b) ∀∃-sentences.*

Proof. All the statements follow from well-known model-theoretic results applied to Theorem 3.14 . By Theorem 4.1.12 in [8], a subclass first-order structures is elementary if and only if it is closed under ultraproducts and elementary equivalence, conditions guaranteed by items (f) and (g) in Theorem 3.14. By Theorem 6.2.5 in [8], an elementary class of structures can be axiomatizable by Horn-sentences if and only if it is closed under reduced products and this condition is assured by 3.14.(f). By Theorem 5.2.6 in

[3] See 11.

[8], an elementary class of structures is $\forall\exists$ axiomatizable if and only if it is closed under direct inductive limits of embeddings; the desired conclusion comes from 3.14.(d). ■

Corollary 3.16 *For each* $G \in \mathbf{RSG}$*, the following are equivalent:*
a) $G \in \mathbf{RSG}_{pp}$.
b) There are a non-empty set I*, a ultrafilter* U *in* I*, a family* $\{G_i : i \in I\} \in \mathbf{RSG}_{fin}$ *and a SG-pure embedding* $f : G \to \longrightarrow(\underset{i\in I}{\to} \prod G_i)/U$.

Proof. (b) ⇒ (a): Since $\mathbf{RSG}_{fin} \subseteq \mathbf{RSG}_{pf} \subseteq \mathbf{RSG}_{pp}$ (by Lemma 3.2), this follows directly from items (a), (b) and (f) in Theorem 3.14.
(a) ⇒ (b): Follows from Lemma 3.2 and 11, because the composition of pure embeddings is a pure embedding. ■

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Hugo Luiz Mariano
Institute of Mathematics and Statistics (IME)
University of São Paulo (USP)
Rua do Matão 1010, Cidade Universitária, CEP 05508-090, São Paulo, SP, Brazil
E-mail: hugomar@ime.usp.br

Francisco Miraglia
Institute of Mathematics and Statistics (IME)
University of São Paulo (USP)
Rua do Matão 1010, Cidade Universitária, CEP 05508-090, São Paulo, SP, Brazil
E-mail: miraglia@ime.usp.br