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A Note on a Liar-type Paradox Sentence and the Incompleteness Theorems

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Abstract

In this short paper, we will demonstrate the construction of a Liar-type sentence based on Curry's Paradox from the Diagonal Lemma style, as well as proofs of the first and second incompleteness theorems from such sentence, highlighting that if the priority of demonstrations is inverted: in this case the first incompleteness theorem follows from the second theorem and the Rosser predicate is not necessary in achieving the result with plain consistency. In the end, we make comments on the use of the Liar Paradox rather than Curry's to construct the undecidable sentence by Kurt Gödel and the philosophical goals involved.

Keywords: Curry Paradox, Diagonal Lemma, Incompleteness Theorems, Liar-type Paradoxes, Plain consistency.

1 Introductory Remarks

In [10], the example 4 of liar-type paradoxes cites a formalized Curry's Paradox sentence, whereby the Diagonal Lemma exists a $\mathcal{L}_A$-sentence ϕ such that $T \vdash \phi \leftrightarrow (Prf_T(\ulcorner \phi \urcorner) \rightarrow f_T(B))$, being $f_T(B)$ a formula which does not contain ϕ. However, the example historically lacks elaboration of variants and the analysis of its consequences, just like the article that introduced the idea of this sentence [11], or a variation from the general Diagonal lemma application to prove Löb's Theorem as was initially develop in [12] with a contemporary approach formalize in [7]. This article proposes performing this task in the simplest and most informative way as possible towards directly to the indemonstrability of consistency of T and as corollary the first incompleteness theorem with plain consistency. Here, we opted mainly for the notations used in (Smith, 2013), with less modifications. Consider the first-order language of arithmetic $\mathcal{L}_A$ and T a recursively enumerable sound extension of Peano arithmetic (PA). n be the numeral for each natural number n. The Gödel number of a formula ϕ by $gn_T(\phi)$ and its numeral by $\ulcorner\phi\urcorner$. The language of propositional logic will be standard, using the propositional variables p, q, r... and the connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$ and $\leftrightarrow$. The provability predicate in T of y, $Prov_T(y)$ is a $\sum_1$-formula where $Prov_T(y) \leftrightarrow \exists x Prf_T(x, y)$. It reads above as "$y$ is the Gödel number of a $\mathcal{L}_A$-sentence provable in T iff exists an x, such that x is the codification's proof of this Gödel number y in the formal system T." To facilitate the visualization of some proofs, we will use the short notation $\Box p$ indicating the predicate of $Prov_T(\ulcorner p\urcorner)$, equivalent to the one used in [3] and ([13], chap. 33). At last, we make use of the Hilbert-Bernays-Löb derivability rules (HBL), represented below:

i. if $T \vdash p$ then $T \vdash \Box p$;

ii. $T \vdash (\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q))$;

iii. $T \vdash (\Box p \rightarrow \Box\Box p)$.

We shall begin with understanding how a variation of Curry-like diagonal lemma can developed and your distinctions to the standard Fixed-Point Theorem.

2 Curry-like Diagonalization

Curry paradox has both conjunctive or syntactic theoretical versions – set-theoretic – and veritative or semantic versions – truth-theoretic. [1] However, as explained in ([10], p. 387) the self-referentiality of a sentence S is summarize as the statement that if S itself is true, then an arbitrary sentence $B*$ follows, that is, propositionally: $(S \equiv S \rightarrow B*)$. If $B*$ is a falsehood, like "$0 = 1$", it leads to inconsistency without any apparent deductive problem. However, the idea of expressing a formalization of the paradox in T, the theoretical-veritative presents itself as the truth predicate, but in T expressed by pure symbolic manipulation in syntactic aspects of the provability predicate $Prov_T$. We could think about how this paradox might be suitable for pushing the system power to extract proof of its own consistency to the limit — if possible — because Curry's sentence significance as a liar-type paradox $p \leftrightarrow (\Box p \rightarrow B)$ allows results without necessarily falling into inconsistencies. Therefore, to formalize

[1] An explanation of the conjunctive theoretical version guided by Russell's Paradox can be seen in [14]. The most publicized and demonstrated version based on the Contraction Principle (where A and B are formulas, then $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)))$ is found in this article too and Curry's canon article [5].

Curry's paradox in T, we must aim to establish formulas: first, to define the consistency guarantee of T if the proof of a diagonalized sentence does not lead us to the proof of a contradiction; and second, a predicate that its definition based on the existence of some formula in which there is the negation of the first function described above.

Consider $diag(gn(\phi))$ as a p. r. function applies in $\ulcorner \phi \urcorner$, having the Gödel number of the diagonalization of ϕ as a result. This can be done by the substitution function in a formula with a free variable its own Gödel number. Take $Diag(y, z)$ as a characteristic function of $diag(y)$ with z as a Gödel number of the diagonalization of y.

Then we have:

Definition 2.1 *Consistent Diagonalization Function:*

$$Gcd(x, y) =_{def} \exists z Diag(y, z) \wedge \neg(Prf_T(x, z) \rightarrow Prov_T(\bot))$$

Definition 2.2 *Curry's Predicate:*

$$C_{Prd}(y) =_{def} \exists x \neg Gcd(x, y)$$

Theorem 2.3 *(Curry-like Diagonal Lemma): $T \vdash C_T \leftrightarrow (Prov_T(\ulcorner C_T \urcorner) \rightarrow Prov_T(\bot))$.*

Proof. By definitions 2.1 and 2.2, we have:

$$C_{Prd}(y) = \exists x \neg \exists Diag(y, z) \wedge \neg(Prf_T(x, z) \rightarrow Prov_T(x));$$

$$\leftrightarrow \exists x \forall z \neg(Diag(y, z) \wedge \neg(Prf_T(x, z) \rightarrow Prov_T(\bot)));$$

$$\leftrightarrow \exists x \forall z \neg(Diag(y, z) \wedge \neg(PrfT(x, z) \rightarrow Prov_T(\bot)));$$

$$\leftrightarrow \forall z(Diag(y, z) \rightarrow ((\exists x Prf_T(x, z) \rightarrow Prov_T(\bot));$$

$$C_{Prd}(y) =_{def} \forall z(Diag(y, z) \rightarrow ((Prov_T(z) \rightarrow Prov_T(\bot))$$

Then, by diagonalizing $C_{Prd}(y)$, we have the sentence C_T, and trivially $T \vdash C_{Prd}(\ulcorner C_{Prd}(y) \urcorner) \leftrightarrow C_T$, so therefore:

$$T \vdash C_T \leftrightarrow \forall z(Diag(\ulcorner C_{Prd}(y) \urcorner, z) \rightarrow ((Prov_T(z) \rightarrow Prov_T(\bot));$$

However, $(Diag(\ulcorner C_{Prd}(y)\urcorner, z)$, z have Gödel number equal to $\ulcorner C_T\urcorner$, then:

$T \vdash C_T \leftrightarrow \forall z((z \leftrightarrow \ulcorner C_T\urcorner) \to (Prov_T(z) \to Prov_T(\bot));$

On the right side of the biconditional, the result leaves us to $T \vdash C_T \leftrightarrow (Prov_T(\ulcorner C_T\urcorner) \to Prov_T(\bot))$.

■

3 Indemonstrability of Consistency and Undecidability

Now we will show that the proofs using Curry's version of the diagonal argument can be done inversely: from the second to the first incompleteness theorem. Consider $Con(T)$ the sentence expressing the consistency of T, which in this case will be symbolized by $\neg\Box(\bot)$, where $\bot$ is a refutable sentence like "$1 = 0$" or "$2 + 2 = 5$".

Theorem 3.1 *(Indemonstrability of Consistency): If T is a consistent theory, then $T \nvdash Con(T)$.*

Proof. So initially, for simplification of theorem 2.3, we have the $\mathcal{L}_A$-sentence: $C \leftrightarrow (\Box(C) \to \Box(\bot))$. With HBL derivability conditions, it follows that:

1. $T \vdash (\Box C \to \Box(\bot)) \to C$, deriving of Theorem 2.3;
2. $T \vdash (\neg\Box C \to C)$, of (1);
3. $T \vdash \Box(\neg\Box C \to C) \to (\Box\neg\Box C \to \Box C)$, applying (i) and (ii) in (2);
4. $T \vdash \Box\neg\Box C \to \Box C$, *modus ponens* in (3);
5. $T \vdash \neg\Box C \to (\Box C \to \bot)$, a tautology;
6. $T \vdash \Box(\neg\Box C \to (\Box C \to \bot)) \to (\Box\neg\Box C \to \Box(\Box C \to \bot))$, applying (i) and (ii) in (5);
7. $T \vdash \Box\neg\Box C \to \Box(\Box C \to \bot)$, *modus ponens* in (6);
8. $T \vdash \Box(\Box C \to \bot) \to (\Box\Box C \to \Box(\bot))$, (ii) in (7);
9. $T \vdash \Box\neg\Box C \to (\Box\Box C \to \Box(\bot))$, (7) and (8);
10. $T \vdash \Box C \to \Box\Box C$, (iii) in C;

11. $T \vdash \Box\neg\Box C \to \Box\Box C$, (4) and (10);
12. $T \to \Box\neg\Box C \to \Box(\bot)$, (9) and (11);
13. $T \vdash C \to (\Box(C) \to \Box(\bot))$, deriving of Theorem 2.3;
14. $T \vdash \neg\Box(\bot) \to (C \to \neg\Box C)$, transformations of (13);
15. if $T \vdash \neg\Box(\bot)$, then;
16. $T \vdash C \to \neg\Box C$, *modus ponens* of (14) and (15);
17. $T \vdash \Box C \to \Box\neg\Box C$, (ii) of (16);
18. $T \vdash \Box C \to \Box(\bot)$, *modus ponens* of (17) and (12);
19. $\neg\Box(\bot) \to \neg\Box C$, contrapositive in (18);
20. $T \vdash \neg\Box C$, *modus ponens* of (19) and (15);
21. $T \vdash \neg\Box\neg\Box C \to \neg\Box(\bot)$, contrapositive of (12);
22. $T \vdash \neg\Box\neg\Box C$, *modus ponens* of (21) and (15);
23. However, $T \vdash \Box\neg\Box C$, applying (i) of (20);
24. $T \vdash \neg\Box\neg\Box C \wedge \Box\neg\Box C$, (22) e (23), here we have a contradiction. *Qua*, $\nvdash \bot$;
25. $T \nvdash \neg\Box(\bot)$, i.e. $T \nvdash Con(T)$.

■

If we look at the simplified proof of the second theorem with the same style offered in [3] it's reduced. It seems due to Curry's version of the diagonal lemma being more weak, i.e., we have less power to derive other sentences than the standard lemma. We can consider the validity of $T \vdash \neg\Box(\bot)$ in (19) sooner than Boolos simple proof. And because of the components of Curry Diagonal Lemma and Theorem 3.1, the First Incompleteness Theorem with plain consistency follows.

Corollary 3.2 *(Curry-like Incompleteness Theorem): If T is consistent, then there is a $\mathcal{L}_A$-sentence C_T such that $T \nvdash C_T$ and $T \nvdash \neg C_T$.*

Proof. Take $T \vdash C_T \leftrightarrow (Prov_T(\ulcorner C_T \urcorner) \rightarrow Prov_T(\bot))$.

If $T \vdash C_T$, then $T \vdash Prov_T(\ulcorner C_T \urcorner) \rightarrow Prov_T(\bot)$;

$T \vdash Prov_T(\ulcorner C_T \urcorner)$ by (i) and by *modus ponens*, $Prov_T(\bot)$. But T is sound, so we have a contradiction, $T \nvdash C_T$. If $T \vdash \neg C_T$, then $T \vdash \neg(Prov_T(\ulcorner C_T \urcorner) \rightarrow Prov_T(\bot))$;

By transformation, $T \vdash Prov_T(\ulcorner C_T \urcorner) \wedge \neg Prov_T(\bot)$, and $T \vdash \neg Prov_T(\bot)$, i.e, $T \vdash \neg\Box(\bot)$. By Theorem 3.1 and T be $\sum_1$-sound, a contradiction. Then $T \nvdash \neg C_T$, being C_T true but not provable in T [2]. ■

4 Discussion and Concluding Remarks

If we understand this result as a certain sense simpler — since the Curry version of the diagonal lemma is weaker and only take account of plain consistency – why didn't Kurt Gödel develop and use other paradoxes? The simple answer is based on that Gödel had distinct goals — besides the fact that the application of diagonalization is a conundrum, until for him. The less committed answer is that the use of the Liar is convenient, because its the most famous, simpler (in the sense of understanding it) and antient paradox known. Maybe the elaboration of a paradox like Curry's didn't occur to him[3]. But Gödel considered that other antinomies could be used, indeed.

Historically, there are three known moments in the literature which Gödel comments on undecidable sentences of paradoxical type: the first is in your canonical article of 1931 when referring to the similarity of his undecidable sentence to the Liar Paradox, he states in a footnote "(...) any epistemological antinomy can be used for an analogous demonstration of non-demonstrability" ([8], p. 231); in ([9] 1934), where Gödel points out the constructing a proposition in which we can apply its own coding, i.e., the diagonal argument, in a paradox-type sentence; and in ([4], p. 26), where the computer scientist Gregory Chaitin claims that managed to demonstrate the first incompleteness theorem based on Berry's Paradox. Gödel only answers that "[...] it

[2]The sentence C could interpreted as "if I am provable, then there is proof of a contradiction in T", a variation of Löb's sentence in ([12], p. 117). It can also interpreted as a disjunction, "there is no proof that a contradiction derives from me (I am indemonstrable) or there is proof of a contradiction in T". As it is indemonstrable, it turns out to be true.

[3]Furthermore, Haskell Curry formulated this paradox more than ten years after Gödel's article in [5]. There are already developing undecidable paradoxical sentences, like the formalization of Yablo's paradox in [1], that with plain consistency we have incompleteness results.

doesn't matter which paradox you use", while Chaitin comments about an informational-theoretical interpretation of incompleteness, giving the impression that the formalization of any antinomy is sufficient to reach the result of undecidability.

However, despite the claims, Gödel was not interested specifically in the formalized paradox per se, but in the results associated with its use in demonstrating the existence of a sentence that could neither be proven nor disproved in some formal systems; even though it was recognize by Carnap later in your *Logische Syntax der Sprache*, the application of the diagonal lemma has relative importance in Gödel's incompleteness results, and aimed at constructing a sentence that expresses its own indemonstrability apparently shows more interest in the incompleteness phenomenon than in the demonstrability of consistency in itself — a hilbertian aim. Could this be the case?

In [15], Solomon Feferman retrace Gödel's steps to incompleteness, and states that his first interests were to prove the consistency of arithmetic and then the relative consistency of analysis. Nevertheless, he realized that the notion of arithmetical truth of sentences would not be definable, moving towards the investigation of undecidable sentences and the notion of demonstrability, previously treated by logicians as a way to formalize truth. This information brings some sense about the use of liar paradox: its natural language version is semantically linked to notions of truth and falsehood. Your achievements, in addition to establishing a strong post argument for the clear distinction between truth and demonstrability, had a major influence of your objectivism about mathematics, and as perceived by Von Neumann, led to the corollary of the indemonstrability of consistency. Although, Gödel was initially interested in the consistency problem and other logical investigations led him to the incompleteness theorem.

In [6], when discussing arithmetic syntax and the diagonal lemma, highlights that Gödel's work is aligned with another logical-philosophical goals, distinct but closed of those mentioned above. Hintikka argues that your results belong to Mathematics, not Logic, claiming that the proof of incompleteness shows that not all arithmetic sentences are true and can be mechanically proved ([6], p. 37). The type of incompleteness exposed by Gödel would be a deductive incompleteness, relative to first-order arithmetic, in which there would not be a computable process to enumerate all its truths. Therefore, incompleteness would not be about the limits of logic, but about the limitation of our idealized notion of computability and mechanical process.

Finally, in the 1931, Gödel exchange letters with Ernst Zermelo and exposes what is called The Master Argument in [13], detailed in [2]. The Gödel's argument is divided into two lemmas, where there is a lemma of expressiveness of the provable sentences in PA and inexpressiveness of the truth sentences in the language of PA. The incompleteness arise when we consider both lemmas. This different way of crafting your argument directly contrasts soundness and completeness, but in a non-constructive manner, deviating from the intention of an "intuitionistically unobjectionable" proof, like in 1931's paper. Thus, Gödel clearly was interested in other topics and sought to elaborate his ideas based on concepts that would help him develop his points. However, this result of Curry paradox version of incompleteness is reasonably interesting, simple and is aligned with Gödel's logical initial hilbertian interests.

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